## Description of the main research directions investigated by the institute

The Institute of Mathematics cultivates the traditional branches of mathematics which Czech mathematicians are famous for, namely the mathematical analysis (differential equations, numerical analysis, functional and abstract analysis), mathematical physics, mathematical logic, complexity theory, combinatorics, set theory, numerical linear algebra, general and algebraic topology, optimisation and control, and algebraic and differential geometry. The researchers are organized in the following six research teams.

### *Abstract Analysis (AA)*

The team studies and classifies mathematical structures, using advanced methods from logic, set theory, and category theory, as well as modern tools from mathematical analysis and algebra. Abstract analysis refers to these areas of science where mathematical logic plays a significant role, even though it is not the main object of study. These areas include descriptive set theory, topology, Banach space theory, and the theory of C* algebras. Further research topics include the operator theory, functional analysis and harmonic analysis.

V. Müller headed the team until the end of the year 2015 when W. Kubiś took the lead. Under the new leadership, the team naturally reassessed the content of its scientific activities. The emphasis shifted from the traditional topics of the theory of Banach spaces, operator theory, and functional analysis to the descriptive set theory, algebraic topology, category theory, and the theory of C* algebras. For this reason, the team changed its original name Topology and Functional Analysis, and since May 2018, it is called Abstract Analysis. Several team members are involved in a new prestigious five-year EXPRO project of excellence funded by the Czech Science Foundation and lead by W. Kubiś. This project aims to explore and classify generic mathematical objects appearing in the mentioned areas of abstract analysis. Apart from this, the team has been successful in the standard grant competition of the Czech Science Foundation.

The team managed to solve several old open problems. Naming just some of them, M. Doucha found an affirmative answer to the Shkarin’s question from 1999 of the existence of separable metric abelian group which contains every separable metric abelian group as its isometric subgroup. J. Kolář answered the question of Simon from 1983 by giving an example of a stationary varifold with non-regular tangent behaviour. M. Engliš with co-authors made an essential step towards the solution of the Arveson-Douglas conjecture from 1998, concerning model theory for commuting tuples of operators. Apart from answering old questions, A. Gogatishvili co-authored a monograph on weighted inequalities involving $\rho$-quasiconcave operators. M. Fabian obtained important results on rich families and Fréchet subdifferentiability. W. Kubiś with his colleagues constructed a universal separable Fréchet space with a graded sequence of semi-norms, introduced the concept of the Katětov functor, and characterized retracts of Fraïssé limits, showing several applications in various areas of mathematics.

### *Algebra, Geometry and Mathematical Physics (AGMP)*

Formed in 2014 upon a bottom-up initiative, this team steadily grows and continuously proves to be one of the most successful within the Institute. Team members investigate algebraic and differential geometry and closely related areas of mathematical physics. Their research focuses on mathematical aspects of modern theoretical physics, mathematical models aiming at understanding the nature of matter, fields, and spacetime. Research topics include representation theory and its applications to algebraic geometry, homological algebra, algebraic topology, applied theory of categories, tensors classification, mathematical aspects of the string field theory, generalized theory of gravitation, and study of Einstein equations.

The team is generally successful in grant competitions of the Czech Science Foundation and received funding for several research projects. Until 2018, members of the team participated in the Albert Einstein Centre for Gravity and Astrophysics, supported by the five-year-long Project of Excellence No. 14-37086G of the Czech Science Foundation. M. Markl, the key member of the team, gained in 2018 the prestigious and generous Praemium Academiae award of the Czech Academy of Sciences, which allowed him to hire several talented postdocs and establish his own ambitious research group.

The team achieved several excellent results in the theory of gravity, analytical solutions of Einstein equations, and modified theories of gravity. Using their conformal-to-Kundt method, V. Pravda and A. Pravdová with their colleagues from the Charles University identified and studied several classes of new static spherically symmetric vacuum solutions of the field equations of modified gravity, including a new non-Schwarzschild black hole with the cosmological constant. This discovery attracted widespread attention and was reported in public media. The group also considerably extended the knowledge of four-dimensional universal spacetimes by determining necessary and sufficient conditions for universality for all Petrov types except type II. Together with M. Ortaggio they proved a large class of electromagnetic fields with vanishing quantum corrections to be exact solutions to virtually any classical non-linear electrodynamics.

Further, M. Markl with his co-authors achieved the ultimate result of the approach he pioneered in the classical 2001 paper on loop homotopy algebras in closed string field theory and constructed the disconnected rational homotopy theory. L. Positselski with his colleagues contributed to the foundations of algebraic geometry by proving the Very Flat Conjecture, essential to understanding the structure of flat morphisms of finite type or presentation and showed the tilting-cotilting correspondence. H. V. Le co-authored a book on information geometry, where she summarized her long-term research on this subject.

### *Differential Equations and Theory of Integral (DETI)*

The team focuses on the qualitative properties of ordinary, generalized, and functional differential equations and systems. Team members investigate mainly the solutions of various boundary value problems, namely their existence, uniqueness, multiplicity, asymptotic behaviour, oscillatory properties, and develop constructive solution methods. The studied equations describe the time evolution of finite-dimensional systems and find primary applications in biology and physics. Analysis of their solutions helps to discover mathematical causalities in real systems, including phenomena like time and space singularities and discontinuous processes. These features are modelled by methods of mathematical and functional analysis employing tools such as the Kurzweil-Henstock integral and time-scales calculus.

The team, namely R. Hakl and his colleagues, obtained important existence and multiplicity results for periodic solutions of singular equations. A. Lomtatidze and J. Šremr developed a new method for constructing upper and lower functions for periodic solutions of second-order Duffing type equations. M. Tvrdý, G. Monteiro, and their colleague from the Charles University published a monograph on the theory and applications of the Kurzweil-Stieltjes integral. A. Lomtatidze wrote an extensive and comprehensive contribution devoted to the existence and uniqueness of a periodic solution to both linear and nonlinear second-order ordinary differential equations with possible singularity in the phase variable, generalizing and complementing the existing theory. Using a new and original method, M. Tvrdý with his co-authors established the existence of the valveless pumping effect for the classical Liebau model. In general, the team achieved many interesting results mainly on the existence, uniqueness, and qualitative properties of solutions of ordinary, functional, and integrodifferential equations.

### *Evolution Differential Equations (EDE)*

Substantial part of the team is involved in the Jindřich Nečas Centre for Mathematical Modeling and in the network for industrial mathematics EU-MATHS-IN.CZ, a part of the European network EU-MATHS-IN.EU. Team members succeeded with several grant applications at the Czech

Science Foundation, but the main achievement was the ERC Advanced Grant MATHEF (Mathematical Thermodynamics of Fluids) of E. Feireisl, which was solved in the Institute for five years until April 2018. Within this ERC grant, E. Feireisl with O. Kreml, V. Mácha, Š. Nečasová, H. Mizerová, and other collaborators built a complete mathematical theory describing the motion of compressible viscous heat-conducting fluids including aspects of stochastic forcing and construction of convergent numerical schemes. Their novel and original approach to the interpretation of the principles of continuum thermodynamics in modelling heat-conducting fluid flow turned out to be a rich source of results for the general theory, as for example the concept of dissipative measure-valued solutions. Results of this project include monographs on the mathematical theory of compressible viscous fluids, singular limits in thermodynamics of viscous fluids, and stochastically forced compressible fluid flows. E. Feireisl with his colleagues further achieved essential results on well-posedness, regularity and stability of the Euler system and similar partial differential equations, proposed a stable finite volume scheme and proved its convergence via dissipative measure-valued solutions.

The EDE team is extra productive and their research is truly collaborative. They took advantage of their wide networks of international partners and jointly investigated many phenomena resulting in a considerable number of important results. Š. Nečasová published a monograph on Navier-Stokes flow around a rotating obstacle and D. Medková a monograph on the Laplace equation. Š. Nečasová with H. Mizerová and other colleagues achieved strong results on global existence for the generalized Peterlin viscoelastic model, global smooth solution of the Cauchy problem for a model of a radiative flow, and the motion of the rigid body in viscous fluid including collisions. The result on the motion of a body with a cavity filled with compressible fluid by G. P. Galdi, V. Mácha, and Š. Nečasová is interesting from the application point of view. O. Kreml, V. Mácha, and Š. Nečasová proved the existence of weak solutions to the full Navier-Stokes-Fourier system with slip boundary conditions in time dependent domains. O. Kreml co-authored the proof of the non-uniqueness of admissible weak solutions to the Riemann problem for the isentropic Euler equations. M. Michálek with colleagues published results on the existence and non-uniqueness of global weak solutions to inviscid primitive and Boussinesq equations and weak solutions of conservation laws and energy/entropy conservation. J. Neustupa with his co-author analysed the structure of the set of stationary solutions to the equations of motion of a class of generalized Newtonian fluids. They provided a spectral criterion for stability of a steady viscous incompressible flow past an obstacle, and contributed to the theory of regularity of weak solutions to the Navier-Stokes equations via one component of velocity and other related quantities. J. Jarušek with his colleagues investigated the solutions of a dynamic contact problem for a thermoelastic von Kármán plate. P. Krejčí and G. Monteiro with co-authors proved a global in time solution for the unsaturated deformable porous media flow with thermal phase transition, thermomechanical interaction, and oscillations of a temperature-dependent piezoelectric rod. M. Kučera with his collaborators studied an influence of unilateral terms to bifurcations of spatial patterns in reaction-diffusion systems.

### *Mathematical Logic and Theoretical Computer Science (MLTCS)*

The research programme of this team concerns mathematical problems coming from theoretical computer science, logic, set theory, finite combinatorics, and control theory. The main topics studied in the team are proof and computational complexity, logical foundations of arithmetic, quantum information theory, theory of graphs, and set theory. The studied questions have foundational importance in itself, but potentially may also have practical applications, for example in data security.

In logical foundations of mathematics, the MLTCS team is one of the world-leading centres of research in bounded arithmetic and proof complexity. Its head, P. Pudlák, has been the principal investigator of the ERC Advanced Grant FEALORA (Feasibility, Logic and Randomness in Computational Complexity) until December 2018. Members of the team are involved in the association of research institutions DIMATIA (Center for Discrete Mathematics, Theoretical Computer Science and Applications) which consists of two Czech universities, the Institute of Mathematics and a network of international partner institutions. Until its formal termination in 2018, team members participated in the research centre of excellence entitled Institute of Theoretical Informatics. Together with the Faculty of Mathematics and Physics of the Charles University, the team participates in the EXPRO project of excellence funded by the Czech Science Foundation and solves several standard projects.

The central topic of the ERC project FEALORA of P. Pudlák was the study of connections between provability and computational complexity. He proved statements about the second incompleteness theorem, established the important role of Herbrand consistency as a TNFP search problem, and addressed the essential question of provably disjoint NP pairs in concrete formal systems. Together with N. Thapen he proved several lower bounds on random resolution and with P. Hrubeš an exponential lower bound on random CNF formulas in the cutting plane proof system. E. Jeřábek proved that the solvability of Diophantine equations in Robinson's arithmetic Q is decidable, more precisely, NP-complete. On top of this, the team members proved that learnability can be undecidable, more specifically, they designed a relatively natural learning problem such that its learnability depends on the validity of the Continuum Hypothesis, and hence is independent of ZFC axioms. Further, they provided the strongest known example of qualitative advantage of quantum communication in computing a function.

T. Masopust, working in the optimal control theory, showed that there is no polynomial algorithm constructing an automata representation of the infimal observable superlanguage of a non-observable language, and developed an exponential algorithm that has a lower algorithmic complexity than existing algorithms for this task. In the theory of graphs, J. Hladký with his co-authors found a common approximate generalization of two famous tree packing conjectures due to Ringel and Gyárfás, proved a slightly weaker version of the most important instance of the Caccetta-Häggkvist Conjecture, and solved the "dense case" of the Loebl-Komlós-Sós Conjecture, which is a classical problem in the area of tree embeddings. Within the set theory, D. Chodounský and B. Kuzeljević proved that there are no P-points in Silver extensions and answered a longstanding question of A. Blass as well as a question of N. Dobrinen and S. Todorčević by demonstrating that there may consistently exist a chain of ultrafilters longer than the size of the continuum, which is strongly increasing with respect to both the Tukey order and the Rudin-Keisler order.

### *Numerical Analysis (NA)*

The team investigates both theoretical and practical aspects of the computational science, mainly numerical methods for partial differential equations and numerical linear algebra. It focuses on questions of convergence, efficiency, and reliability of numerical methods for partial differential equations including matrix computations and high-performance implementations on parallel computer architectures. Members of the team are experts in the finite element method, saddle-point systems of linear algebraic equations, domain decomposition methods, high-performance computing and computational fluid dynamics.

Members of the team are involved in the Jindřich Nečas Center for Mathematical Modeling and in the network for industrial mathematics EU-MATHS-IN.CZ, part of the European network EU-MATHS-IN. Apart from several standard projects of the Czech Science Foundation, the team succeeded in competitions for the CPU time at large European computers. T. Vejchodský received a Neuron Impuls grant funded by the private Neuron Endowment Fund.

The following selection of results represents the research of the team during the evaluated period. M. Rozložník published a monograph on saddle-point systems of linear algebraic equations and methods for their iterative solution. With his co-authors he also achieved one of the first results on the numerical behaviour of the GMRES method for solving singular linear systems and published a critical analysis of the numerical stability of the pipelined conjugate gradient method. T. Vejchodský used flux reconstructions to derive robust and straightforward a posteriori error bounds for reaction-diffusion systems, and used similar flux reconstructions to compute guaranteed lower bounds on eigenvalues of elliptic operators by the Lehmann-Goerisch method. P. Kůs and J. Šístek tackled the challenging problem of how to parallelize mesh adaptive algorithms. J. Šístek with his colleagues successfully applied the domain decomposition method for large-scale simulations of subsurface water flows. J. Calvo developed a new approximation of a virtual coarse space for domain decomposition methods reducing the resulting condition number and resulting computational time. M. Křížek with his co-authors achieved new tight bounds on angle sums of nonobtuse simplices having essential applications in the finite element method. B. She with H. Mizerová, a colleague of the EDE team, constructed a new conservative scheme for the Fokker-Planck equation describing the behaviour of viscoelastic polymeric fluids.

## Research activity and characterisation of the main scientific results

Research activities of the team are described below, divided into main topics.

***Generic mathematical structures***
(Bartoš, Doucha, Ghasemi, Kąkol, Kubiś)

This is an important topic, stemming from model theory, currently studied intensively by several research groups (e.g. in Dresden, Münster, Novi Sad, Lyon, Paris, Toronto), in particular due to its applications in the theory of Polish groups and topological dynamics. Concrete generic structures are called *Fraïssé limits,* which could be described as the most complicated objects (within a certain class) that at the same time are highly symmetric (in that they have rich automorphism groups). Fundamental examples are the random graph, the set of rational numbers, and the universal homogeneous metric space of Urysohn.

A significant part of the research within the evaluation period was devoted to studying and exploring generic structures in category theory, often enriched over metric spaces. The main research within this topic was led by Kubiś, an internationally recognized expert in the area.

Results included in the preprint of Kubiś titled *Metric-enriched categories and approximate Fraïssé limits* [arXiv:1210.6506] contain fundamental tools for studying generic structures in various areas of mathematics. This is a novel approach, capturing mathematical objects both inside and outside the model theory, contrary to what was done before. Another (submitted) work *Weak Fraïssé categories* [arXiv:1712.03300] contains a more general framework, relaxing the crucial assumption called the *amalgamation property*.

Kubiś obtained in [Forum Math., 2015, 0443127] an important characterization of retracts of Fraïssé limits, in the framework of category theory. Retractions are of crucial importance in the study of endomorphism semigroups. The results provide new insights into the structure of endomorphisms of Fraïssé limits and have several applications e.g. to metric spaces, Banach spaces and linear orderings.

One of the most important features of generic objects is the rich structure of their automorphism groups. In particular, the automorphism group of the Fraïssé limit is often universal in the sense that it contains copies of all automorphism groups of structures in the class under consideration. The work [Appl. Categ. Struct. 2017, 0475997] of Kubiś and Mašulović unifies several results on the universality of automorphism groups of Fraïssé limits, showing that all of them follow from the existence of a special functor (called the *Katětov functor*) building the Fraïssé limit. Among applications, some surprising semigroup-theoretic properties of the endomorphism monoid of Fraïssé limits were proved. Furthermore, Kubiś and Shelah [arXiv:1811.09650] constructed several new Fraïssé limits whose automorphism groups are not universal, in particular, showing that the corresponding classes do not admit any Katětov functor.

Another significant line of research within the topic of generic structures is discovering existing mathematical objects as Fraïssé limits or constructing new ones in order to show that there are universal structures in a given class. For example, the work of Kubiś and Kwiatkowska [Rev. R. Acad. Cienc. Exactas Fís. Nat., 2017, 0477378] presents the Lelek fan and the Poulsen simplex as Fraïssé limits, providing the first direct and short proofs of their uniqueness. The authors develop a new geometric description of smooth fans and Choquet simplices. Furthermore, the article of Bargetz, Kąkol, and Kubiś [J. Funct. Anal., 2017, 0469673] contains a new construction of a universal graded Fréchet space, where the crucial tool was the generic operator on the Gurarii space (the Fraïssé limit of finite dimensional normed spaces) developed earlier by Garbulińska-Węgrzyn and Kubiś. A natural extension property makes this space unique up to suitable isometries.

Within the framework of this topic, there is also ongoing research dealing with other generic structures: $C^*$-algebras and compact connected metric spaces. Namely, Ghasemi and Kubiś [arXiv:1903.10392] found a non-commutative analog of the Cantor set, a generic separable AF-algebra that additionally admits a surjective homomorphism onto every other separable AF-algebra. Such a $C^*$-algebra was not known before. This work was done mostly in 2018–2019 and the final version has already been accepted for a publication in the Journal of Functional Analysis. Other ongoing work (Bartoš, Kubiś) deals with generic compact metric spaces, in particular, the pseudo-arc, pseudo-circles and similar geometric objects.

Doucha used the methods of Fraïssé theory to construct several universal metric groups whose existence had been an open problem. In [Trans. Am. Math. Soc., 2017, 0474420] he answered a question of Shkarin and constructed a separable metric abelian group which isometrically contains all separable metric abelian groups as subgroups. In [Trans. Am. Math. Soc., 2019, 0512066] with Malicki, they obtained several new results, and improved and simplified proofs of several older ones, on generic representations of countable discrete groups in several countable and separable metric structures. This includes answering a question of Kechris and Rosendal. Additionally with Valette in [J. Group Theory, 2019, 0498908], they connected property (T) of Kazhdan with generic unitary representations.

In [Monatsh. Math., 2019, 0504468], Doucha also used the methods of Fraïssé theory to construct a universal object in the category of actions of countable locally finite groups on separable metric spaces and he proved that such a universal object cannot exist for actions of groups that are not locally finite.

***Descriptive set theory and infinite combinatorics***
(Doležal, Doucha, Kubiś, Kurka)

A fruitful area of research in this topic are connections with the theory of Banach spaces. In the paper [Isr. J. Math., 2019, 0505498] that solves a problem posed by Godefroy, Kurka proved that the class of spaces isomorphic to the space $c_0$ is not Borel in the Effros Borel structure of separable Banach spaces. This means that, in general, it is quite difficult to determine if a given separable Banach space is isomorphic to $c_0$. Another consequence of the methods provided is, for instance, the same result for the class of spaces with an unconditional basis. So, in the same spirit, it is quite difficult to determine if a given separable Banach space has an unconditional basis.

The main tool of the proof is a modification of the remarkable construction by Argyros, Gasparis and Motakis, itself based on the famous construction of Bourgain and Delbaen. In addition, it turns out that the methods are connected to infinite combinatorics, as one can apply Pták's combinatorial lemma to show the key property of the constructed space.

The main idea behind descriptive set theory is the study of certain systems of “nice” subsets of Polish spaces (e.g. of the real line). It is one of the most substantial areas of research in set theory and its strength lies mostly in the fact that it has plenty of applications in various other areas of mathematics. The research of our group is oriented mostly toward such applications, mainly in group theory or Banach space theory.

One of the most important results of the team from this area is [Arch. Math. Logic, 2016, 0463607]. The main result of this paper is a certain dichotomy concerning the existence of independent sets with respect to countably many relations. While this result seems to be a little technical, several interesting applications in group theory are provided. Namely, it is shown that in many important cases, a given group has a subgroup having some nice properties, which is generated by a perfect set.

The importance of descriptive set-theoretic tools can also be nicely seen in our research concerning Banach space theory. We are mostly interested in the study of the relations “being isometric” or “being isomorphic”. Instead of proving or disproving that a certain isometry or isomorphism exists, we rather look at the equivalence classes of individual Banach spaces and compare their complexities. This is surprisingly a very useful technique in this setting, which often leads to strong results. This direction of study is taken by Cúth, Doležal, Doucha, and Kurka in the preprint [arXiv:1912.03994]. As a starting point, they prove that the infinite-dimensional Hilbert space is characterized as the unique separable infinite-dimensional Banach space whose isometry class is closed, and also as the unique separable infinite-dimensional Banach space whose isomorphism class is $F_\sigma$. In a similar way, the complexities of isometry classes of several other classical Banach spaces are provided as well.

Another connection with functional analysis and metric geometry was made by Cúth, Doucha and Kurka in [arXiv:1804.11164] and [arXiv:2004.11752]. The motivation for their research is as follows. The study of analytic/Borel equivalence relations and their mutual reducibility is one of the main research topics of current descriptive set theory. While this theory finds many applications e.g. in universal algebra, topology, and other disciplines with a strict notion of isomorphism, it is not always sufficient in "metric subjects" such as metric geometry and functional analysis, where the isomorphism relation is often continuously approximated (e.g. Banach-Mazur distance approximates linear isometry and Gromov-Hausdorff distance approximates isometry). The authors develop a theory of analytic/Borel pseudometrics that generalizes the standard theory of equivalence relations. Moreover they show that standard distances such as the Gromov-Hausdorff, Banach-Mazur, Kadets distance, etc., are mutually bi-reducible.

Doležal, together with Vejnar, obtained in [Topology Appl., 2015, 0442124] a complete classification (under the assumption of projective determinacy) of the Borel–Wadge complexity of the space of bounded continuous functions on a projective metrizable space $X$, depending on the complexity of $X$. The Borel–Wadge hierarchy is one of the most interesting hierarchies studied in descriptive set theory. While an analogous classification was previously known for the space of all (not necessarily bounded) continuous functions, this case turned out to be much more complicated.

Doležal is further a coauthor of several papers dealing with certain notions of smallness in Polish groups or metric spaces in general. Together with Preiss and Zelený he studied in [Isr. J. Math., 2016, 0465182] so called *sigma-porous sets*. Using an infinite game (one of the key tools in descriptive set theory), they proved that in many important cases, if a set is not sigma-porous then it contains a compact subset which is also not sigma-porous. Another notion of smallness studied by Doležal, Rmoutil, Vejnar, and Vlasák is called Haar meagerness. In [J. Math. Anal. Appl., 2016, 0458946] many known facts from abelian Polish groups were extended to general Polish groups. Among other results, they showed that some Polish groups (e.g. the real line) can be decomposed into a Haar meager set and a Haar null set. In [J. Math. Anal. Appl., 2017, 0466160], Doležal and Vlasák explained that, in contrast to the classical notion of meager sets, there are Haar meager sets which are not contained in any $F_\sigma$ Haar meager set. While Doležal is the only coauthor of these papers who is a member of this team, his contribution to the results was relatively high (for example, the decomposition result mentioned above is mostly due to him).

Other applications of descriptive set theory (namely of infinite games) are described by Doležal and Moors in [Topology Appl., 2017, 0481511]. These applications include sufficient conditions for the product of two topological spaces to be a Baire space, for a semitopological group to be a topological group, or for a separately continuous function to be continuous at the points of a certain large set.

***Banach space theory***
(Fabian, Hájek, Kąkol, Kania, Kubiś, Kurka)

This topic, especially the non-separable case, has a long tradition here, originated by important works of Fabian, Hájek, John, Pelant, and Zizler (the last three were former team members), culminating in several monographs, including the one by M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler [*Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, Springer-Verlag, New York 2011, 0358155] serving both as a valuable reference for researchers and as an excellent textbook for higher level students. Within the evaluation period, the team members worked both on separable and non-separable Banach spaces.

One important line of research was devoted to Banach spaces with a rich structure of projections. One of the best structures is a *projectional skeleton* (a notion introduced by Kubiś in 2009) which is closely related to the existence of a rich family of complemented separable subspaces.

The works of Cúth and Fabian [Proc. Am. Math. Soc., 2015, 0438336], [J. Math. Anal. Appl., 2017, 0468148], [J. Funct. Anal., 2016, 0455101], and Fabian and Montesinos [Funct. Approximatio, Comment. Math., 2018, 0498482] study projectional skeletons with some extra properties, characterizing several known classes of Banach spaces. A related older concept of a rich family is particularly useful for separable reductions, namely, proving that a certain property holds if it is valid in every closed separable subspace. The property could be, for example, differentiability of a fixed function defined on a large Banach space.

Within this line of research, Fabian (jointly with Ioffe) succeeded in obtaining separable reductions of some statements concerning Fréchet subdifferentials; the results are contained in [Set-Valued Var. Anal., 2013, 0422304], [J. Convex Anal., 2016, 0462793]. Furthermore, the work of Fabian, Ioffe, and Revalski [Proc. Am. Math. Soc., 2018, 0495242] proves a separable reduction (via rich families) of an important functor calculating the so called *regularity of mapping*s. Actually, the above-mentioned work of Cúth and Fabian [J. Funct. Anal., 2016, 0455101] also deals with separable reductions of Fréchet subdifferentials.

Other lines of research within this topic included: Establishing quantitative constants for separation in various Banach spaces and obtaining strengthenings of classical results, such as Kottman's theorem in the symmetric setting (Hájek, Kania, Russo [J. Funct. Anal., 2018, 0494452]); mixing isometric and isomorphic aspects of Banach space theory by proving geometric results invariant under renormings (Castillo, González, Kania, Papini [arXiv:1910.01626], accepted for publication in Proceedings of the American Mathematical Society); establishing the existence of uncountable Auerbach systems in sufficiently large Banach spaces by using infinitary combinatorics (Hájek, Kania, Russo [arXiv:1803.11501], accepted for publication in Transactions of the American Mathematical Society).

In the last couple of years, Hájek's research focused on biorthogonal systems in Banach spaces, mostly non-separable ones. Among his results is a construction of a long Auerbach system in a weakly Lindelöf determined Banach space, for density larger than omega one, while for density omega one, there is a counterexample using the continuum hypothesis (to appear in Trans. Am. Math. Soc.).

### *Operator theory, complex analysis*

(Engliš, Müller)

This is a broad topic including applications of methods of functional analysis and complex analysis to operator theory (in particular, multi-variable operator theory), complex geometry or mathematical physics.

In the area of multi-variable operator theory, an important step was made in [Adv. Math., 2015, 0441811] towards a solution of the so-called Arveson-Douglas conjecture: In a highly influential paper W. Arveson: *Subalgebras of $C^*$-algebras III*, Acta Math. 181 (1998), 159–228, the author laid down a theory of models for commuting tuples of operators on a Hilbert space (so-called *row* *contractions*). An important piece missing in this theory is the question of commutativity of the model operators modulo the compact operators, which eventually became known as the Arveson-Douglas conjecture. Despite considerable efforts by many authors, in its most general form this conjecture remains open to this day, with only some special cases settled. The current article proves the conjecture, though not yet in the utmost generality (the conjecture can be restated as an assertion about ideals in certain Hilbert modules of vector-valued functions, and the article gives a proof for the case of radical ideals of scalar-valued functions). Furthermore, as the main ingredient of the proof, the article develops machinery which is totally new in the field and has potential for applications to other problems. Although falling short of proving the full conjecture, the paper has been described by O. Shalit as „the biggest step forward on this problem since 2008“ (in his informal announcement from the Oberwolfach workshop where the proof was first presented, see Souvenirs from the Black Forest); since its publication in 2015, the article has already earned 11 citations in follow-up papers. The journal itself (Advances in Mathematics) is one of the top mathematics journals in the world with a long tradition.

The connections with complex geometry relate mostly to the existence of so-called balanced metrics on complex domains; these are Kähler metrics whose potential, loosely speaking, equals the reciprocal of their associated reproducing kernel. Their existence for compact manifolds was proved by the Fields laureate S. Donaldson in 2001, however their existence on complex domains remains an open problem to this day (and their uniqueness remains an open problem even on the unit disc). Three papers in this direction were published by Engliš (two of them in collaboration with Bommier-Hato and Youssfi from Marseille).

The connections to mathematical physics concern mostly the use of Toeplitz and similar operators for quantization, together with the related problem of high-power asymptotics of reproducing kernels of holomorphic line bundles, as well as links to the noncommutative geometry of A. Connes, including results about the Dixmier trace of the associated operators. Contributions in this area include investigations of Engliš of spectral triples constructed using Toeplitz operators (with Iouchum and Falk from Marseille), of high-power asymptotics of harmonic Bergman kernels, of function spaces arising as kernels of higher powers of the Cauchy-Riemann operators (following an idea due to Boutet de Monvel and Guillemin) and their operators, and a resolution of the decades-old problem of characterizing Hankel operators on the Hardy space that lie in the Dixmier ideal (with G. Zhang).

Yet another thread in this topic concerns analysis on Hermitian symmetric spaces, especially bounded symmetric domains; here methods of representation theory of Lie groups and/or Jordan triples become an important tool. Outputs in this area include work due to Engliš (joint with Upmeier) [Trans. Am. Math. Soc., 2015, 0434137] proving the existence of an asymptotic expansion of the Berezin transform and of the Peter-Weyl decomposition of the Toeplitz quantization on general (complex, real, compact or noncompact) symmetric spaces. Furthermore, unlike previously existing approaches, machinery is developed to treat all these cases in a completely uniform manner, thereby demonstrating the inherent algebraic relationships.

The research in operator theory was concentrated mainly on the study of joint numerical ranges and their applications to various problems of analysis.

The numerical range of Hilbert space operators is a classical concept introduced by Hausdorff and Toeplitz in 1918. The most important property of the numerical range of a single operator is its convexity. The numerical range can be naturally defined also for $n$-tuples of Hilbert space operators, however, the joint numerical range of an $n$-tuple of operators is in general not convex any more. Because of this obstacle, the joint numerical range is much less well understood than the numerical range of a single operator.

The study of the joint numerical range of $n$-tuples of Hilbert space operators and their applications was the main topic in a series of 3 joint papers of Müller with Tomilov.

In [J. Funct. Anal., 2018, 0483702] Müller with his co-author Tomilov found a new theorem describing large subsets of the joint numerical range of an operator tuple of commuting bounded operators on a Hilbert space in terms of the joint spectrum of the tuple. This result allowed them to prove a criterion for a Hilbert space operator to have a unit circle in its spectrum in terms of geometric properties of operator orbits. The criterion is a far-reaching extension of the analogous well-known result due to Arveson concerning just unitary operators. Moreover, the paper's methods led the authors to the discovery of new effects in convergence properties of operator orbits. In particular, they showed that, under natural spectral assumptions, weak convergence of powers of a Hilbert space operator imply the uniform convergence of their compressions to an infinite-dimensional subspace. Existing results of the same flavour concerned a very particular situation of single orbits of unitary operators.

In [J. Lond. Math. Soc., 2019, 0501201] Müller with his co-author Tomilov obtained a new result identifying large parts of the joint numerical range of an operator tuple on a Hilbert space by means of the essential numerical range of the tuple and its joint point spectrum. Moreover, they found a unified approach to the description of several joint numerical ranges of tuples of Hilbert space operators, and obtained several new characterizations of some of these numerical ranges. By developing and refining their recent technique for the study of operator iterates, they revealed further fine properties of Hilbert space operators with powers weakly converging to zero. In particular, they proved that the compressions of powers of such an operator well approximate the powers of any strict contraction on an appropriate infinite-dimensional subspace. This is the first result of this kind in the literature.

In [Trans. Am. Math. Soc., 2019, 0507774] Müller with his co-author Tomilov provided an innovative methodology for constructing diagonals of Hilbert space operators based on the so-called *Blaschke-type con*dition, also appearing in the literature for the first time. They recognized an important pattern in results about diagonals of operators by noting that a sufficient condition for a sequence to be the diagonal of some operator is that the sequence does not approach the boundary of the essential numerical range too rapidly. They turned the pattern into powerful theorems which apply to the vast majority of operators, and provided sufficient conditions for a sequence to be in the set of diagonals (not only in the closure of this set, as in many previously known results). Moreover, their new tools allowed them to obtain their results in the more general and demanding setting of operator tuples, and to prove several related results, including, in particular, theorems on diagonals of perturbations of operator tuples. Finally, their new technique led to several essential generalizations of results on constructing operator-valued diagonals of Hilbert space operators. These results extend a number of old statements in the literature, most notably those due to Herrero, Stout, and Bourin.

Further research was concentrated in linear dynamics, ergodic theory, dilation theory and spectral theory in max algebras.

Linear dynamics is a rapidly developing area of operator theory with close relations to the theory of dynamical systems. Linear dynamics studies especially orbits that behave in a very irregular, chaotic way. The basic concept is the notion of hypercyclic vectors (vectors with dense orbits) and its variants (supercyclic vectors, mixing, weakly mixing, frequently hypercyclic vectors etc.). The notions of Li-Yorke chaos and distributional chaos were systematically studied by Müller and co-authors (Bernardes, Bonilla, Peris) in [Ergodic Theory Dyn. Syst., 2015, 0446406]. These notions play an important role in the theory of dynamical systems and the paper introduced them to the field of linear dynamics.

Further research concerned the mean ergodic theorem for power bounded Hilbert space operators with respect to subsequences. There are many results of this type for unitary operators or for Hilbert space contractions. The method developed for the much larger class of power bounded operators is quite new. In fact, it is so new that it has not attracted the deserved attention of specialists working in ergodic theory yet.

***Function spaces***
(Gogatishvili, Musil)

Function spaces have been a central part of functional analysis since the 1950s and they have seen a true renaissance ever since the early 1990s. Their indispensable role in solving difficult problems appearing in mathematical physics, PDE's etc., is widely known. Recently it seems that their importance is increasing dramatically in connection with new tasks. We have seen on several occasions that a choice of appropriate function space led to a decisive breakthrough for a specific problem.

Among the most important problems within this topic studied by Gogatishvili, several characterizations of Sobolev spaces over Euclidean spaces of even order were obtained using the differences between a function and the ball average of a function. These results may shed new light on the theory of higher-order Sobolev spaces over spaces of homogeneous type. Several characterizations of Besov and Triebel-Lizorkin spaces using the differences between a function and the ball average were given together with corresponding results for inhomogeneous Besov and Triebel-Lizorkin spaces. These results, introducing Besov and Triebel-Lizorkin spaces with any smoothness of order between 0 and 2 over spaces of homogeneous type, were established in [Nonlinear Anal., Theory Methods Appl., 2015, 0446760] and [J. Math. Anal. Appl., 2016, 0447568]. Gogatishvili also proved necessary and sufficient conditions for continuous embeddings of Sobolev-type spaces modelled upon rearrangement-invariant Banach function spaces into generalized Hölder-type spaces defined by employing the -modulus of smoothness in [Ann. Mat. Pura Appl., 2015, 0442892].

The $s$-numbers of weighted Hardy-type operators acting between Banach function spaces were also studied in [J. Approx. Theory, 2016, 0458214]. Under some geometric assumptions on the space and the weight functions, two-sided estimates were obtained for its approximation, isomorphism, Bernstein, Gelfand and Kolmogorov $s$-numbers.

In [Bull. Sci. Math., 2016, 0459257], some norm estimates of maximal multiplier operators were studied in variable-exponent-Lebesgue spaces, considering the case when the multiplier is a Fourier transform of a compactly supported Borel measure.

A criterion for precompactness of sets in Banach function spaces and Lebesgue spaces was extended to the setting of quasi-Banach function spaces and so-called power quasi-Banach spaces, respectively, in [Proc. R. Soc. Edinb., Sect. A, Math., 2016, 0460329]. These criteria were applied to establish compact embeddings of abstract Besov spaces into quasi-Banach function spaces and illustrated on embeddings of Besov spaces into Lorentz-type spaces.

Gogatishvili wrote a book, where he studied a class of $\rho$-quasiconcave functions in their full generality. These results are needed for a comprehensive study of weighted inequalities involving the class of $\rho$-quasiconcave functions. He illustrated these results on weighted inequalities of Hardy type, including those involving suprema, and on reverse forms of such inequalities [World Scientific, 2018, 0491680].

Moreover, Gogatishvili proved a sharp estimate for the $k$-modulus of smoothness, modelled upon a Lebesgue space. This sharp estimate was used to find necessary and sufficient conditions for continuous embeddings of Sobolev-type spaces into generalized Hölder spaces defined through the $k$-modulus of smoothness. General results were illustrated in examples. In particular, a generalization of the classical Jawerth embedding was given in [J. Funct. Anal., 2019, 0497233].

Gogatishvili computed a $K$-functional related to some spaces like small, or classical Lebesgue or Lorentz-Marcinkiewicz spaces. This computation allows one to determine the interpolation space in the sense of Peetre. It follows that the resulting spaces are always $G\Gamma$-spaces, which cover many customary function spaces. One of many motivations of this research was to obtain a regularity estimate for the so-called very weak solution of a linear equation in a domain with data in the space of integrable functions with respect to the distance function to the boundary of the domain (see [Differ. Equ. Appl., 2018, 0486954], [Nonlinear Anal., Theory Methods Appl., 2018, 0496179] and [J. Math. Anal. Appl., 2020, 0510259]).

Gogatishvili also focused on the boundedness of general quasi-linear operators in weighted Lebesgue spaces restricted to the cone of quasi-concave functions. He completely solved several open problems in this field in [Proc. R. Soc. Edinb., Sect. A, Math., 2020, 0522215]. He also obtained a characterization of embeddings between Lorentz-type spaces defined by two different weighted means in [J. Func. Anal., 2017, 0477928].

### ***Geometric measure theory***

(Kolář)

One of the main methods for solving systems of partial differential equations is the Direct method of the Calculus of variations, when a minimizer of a functional (or its stationary point) is found in a suitable class of objects. The class has to be large enough to have required compactness properties, which allows for both so-called *non-parametric* and *parametric* settings. When the class is enlarged, there is need of an additional step proving the "regularity" of the minimizer. For example, for the surface area functional, a suitable class consists of generalized surfaces called *varifolds*. The surface area functional corresponds to a minimal surface system, which is an important case of a quasilinear elliptic differential equation. The regularity task then starts with the question of uniqueness of tangents (which roughly corresponds to the existence of derivatives of functions in the non-parametric case).

These are, however, open questions with positive results mostly having exceptional sets of a certain dimension.

In [Calc. Var. Partial Differ. Equ., 2015, 0448479] Kolář answered an open question of Simon (1983) by giving an example of a stationary varifold with non-regular tangent behaviour. Simultaneously he gave the first rectifiable example for a related question of Allard (1972).

## Research activity and characterization of the main scientific results

The core of the team is formed by members of the Department of Algebra, Geometry, and Mathematical Physics which was established in 2014 to bring together researchers and students working in these fields. The team focuses on excellent results. Ten of the papers oriented on mathematical physics were published in the most prestigious journals covered by Nature index (which contains less than 5% of the journals on natural sciences indexed in the Web of Science) – eight of them in Journal of High Energy Physics, one in Physical Review Letters, and one rapid communication in Physical Review D. Most of the other papers were also published in the leading journals in the field including Advances in Mathematics and Communications in Mathematical Physics. Let us proceed with the discussion of the research areas studied by the team.

### *Mathematical relativity*

Topics falling, in general, on the interface between (Lorentzian) differential geometry and mathematical physics, are studied by V. Pravda, A. Pravdová, M. Ortaggio, T. Málek, I. Khavkine (from December 2017), and M. Kuchynka (Ph.D. student).

One of the principal research topics of the mathematical relativity group is the search for exact solutions to field equations of generalized theories of gravity. In general, field equations of modified theories of gravity are highly non-linear and substantially more complicated than the Einstein equations. For this reason, very few exact solutions of these theories are known and researchers often resort to numerical methods to continue their studies. Nevertheless, similarly as in Einstein's gravity, exact solutions represent a fundamental tool for our understanding of various mathematical and physical aspects of the theory. Exact solutions also often serve as toy models to test various hypotheses and as testbeds for numerical codes. One can search for various exact solutions of a specific modified theory of gravity. However, this approach often leads, as can be seen in the literature, to rediscovering, again and again, the same classes of metrics, only in the context of distinct modified gravities. To avoid this redundancy, it is thus perhaps more effective to start with a study of Lorentzian metrics which simultaneously solve all (or at least a large class of) vacuum field equations of modified gravities and in this sense, they are exceptional solutions to the Einstein equations "immune" to all possible correction terms added to the Einstein field equations. We call spacetimes described by these solutions Universal spacetimes.

*Universal spacetimes and related topics*

V. Pravda and A. Pravdová, in collaboration with S. Hervik (Stavanger University, Norway), started a systematic study of necessary and sufficient conditions for the universality of Lorentzian spacetimes (of arbitrary dimension) already in 2014. They have found necessary and sufficient conditions for type N spacetimes and partial results for type III spacetimes [Class. Quantum Grav., 2014, 0433091]. Together with T. Málek, they studied universality for type II and D spacetimes and presented explicit examples of such spacetimes for all composite number of dimensions [Class. Quantum Grav., 2015, 0452809]. Furthermore, they have proved that type II and D universal spacetimes do not exist in five dimensions. The existence of such spacetimes in prime number dimensions greater than five remains open. In [J. High Energy Phys., 2017, 0478962], S. Hervik, V. Pravda, and A. Pravdová focused on the case of four dimensions, where more complete results can be obtained. In fact, they have found necessary and sufficient conditions for universality for all Petrov types except type II. By definition, all universal spacetimes are Einstein spaces. To proceed beyond, they studied various non-Einstein generalizations of universal spacetimes. In [Phys. Rev. D, 2019, 0500815], M. Kuchynka, T. Málek, V. Pravda, and A.

Pravdová studied so-called almost universal spacetimes. For algebraic types N and III, they have found sufficient conditions for "almost universality" and demonstrated on several examples how to construct exact solutions of various higher-order gravity theories using almost universal spacetimes.

In a series of papers, members of the team have also extended the notion of universality to theories of generalized electrodynamics. First, in [Class. Quantum Grav., 2016, 0459163], M. Ortaggio and V. Pravda have determined the class of $p$-forms possessing vanishing scalar invariants at arbitrary order in $n$ dimensions. They employed these results in [Phys. Lett., B, 2018, 0487352], where they have proved that a large class of electromagnetic fields is immune to any modifications of Maxwell's equations in the form of powers and derivatives of the field strength, thus being exact solutions to virtually any classical non-linear electrodynamics. Such corrections appear, e.g., in effective theories motivated by quantum electrodynamics or the string theory. Subsequently, M. Kuchynka and M. Ortaggio have extended these results to include the coupling of non-linear electrodynamics with gravity [Phys. Rev. D, 2019, 0502155] where they have obtained a full characterization of Einstein-Maxwell $p$-form solutions in arbitrary dimensions, for which all higher-order corrections vanish identically. Such "universal" configurations thus simultaneously solve a large class of Lagrangian theories including both modified gravities and modified electrodynamics. Subsequently, M. Kuchynka has extended these results to include Yang-Mills fields [Class. Quantum Grav., 2019, 0509629]. Recently also some members of the team started to investigate possibilities of connecting the topic of universal spacetimes with black hole solutions of modified gravities. First results in this direction on "universal black holes" were published by S. Hervik and M. Ortaggio in 2020 [J. High Energy Phys., 2020, 0522119].

*Black holes in quadratic gravity*

Quadratic gravity is a modified gravity where quadratic terms in curvature tensors are added to the Einstein-Hilbert action linear in curvature. In some sense, quadratic gravity is the most natural simple generalization of Einstein's gravity, which is the "first approximation" of quadratic gravity. Interestingly, the Schwarzschild black hole is an exact solution of both theories – Einstein's gravity and quadratic gravity. Recently, [Lu et al, Phys. Rev. Lett., 2015] showed that in contrast to Einstein gravity, in quadratic gravity, another spherically symmetric and static black hole exists and thus uniqueness is lost. Due to the complexity of the field equations, this was done in part using numerical methods. Since this work was closely related to an ongoing research project of V. Pravda and A. Pravdová in collaboration with J. Podolský and R. Švarc (Charles University, Prague), they started a thorough investigation. In [Phys. Rev. D, 2017, 0474091], they have pointed out that all Robinson-Trautman spacetimes are conformal to Kundt spacetimes. Note that all static, spherically symmetric spacetimes (and thus also the new black hole) belong to the Robinson-Trautman class. Furthermore, under an appropriate assumption, all correction terms in quadratic gravity can be combined in the so-called Bach tensor, which is conformally well-behaved. Thus, they expected that looking at the extremely well-studied problem of spherically symmetric spacetimes from a different point of view (spacetimes conformal to Kundt) could lead to a dramatic simplification of the field equations of quadratic gravity. In [Phys. Rev. D, 2018, 0491681] and [Phys. Rev. Lett., 2018, 0497779] they obtained such equations in the case without and with cosmological constant, respectively, and due to the much simpler form of the field equations, they were able to study the corresponding solutions, including black holes, using analytical methods, which has led to more robust results and clearer understanding. Furthermore, since the work [Lu et al, Phys. Rev. Lett. 2015] assumes vanishing cosmological constant, in [Phys. Rev. Lett., 2018, 0497779], they have also presented a generalization of the previously discussed black hole to the case of a non-vanishing cosmological constant. Both these publications are in the form of brief letters. The corresponding detailed paper in the case of vanishing cosmological constant has been published in 2020 [Phys. Rev. D, 2020, 0520855] and a full publication of the case with a non-vanishing cosmological constant is under preparation.

*Other selected results in mathematical relativity*

In [Class. Quantum Grav., 2019, 0508337], I. Khavkine presented the first practical and systematic method of constructing a provably complete set of local linear gauge-invariants in linearized gravity on a large class of backgrounds of sub-maximal symmetry. As a demonstration of its practicality, this method was then applied to compute such complete sets of gauge invariants for cosmological FLRW backgrounds (equivalent to some earlier but ad-hoc results of Khavkine and co-authors) and spherically symmetric Schwarzschild-Tangherlini black hole backgrounds (new results). Previously, a systematic construction was known only for the maximally symmetric case (Minkowski, (anti-)de Sitter spacetimes). This 2019 paper has been already quoted in Comm. Math. Physics and the Arxiv preprint in Phys. Rev. Letters.

I. Khavkine, in collaboration with A. Garciá-Parrado, has given in [J. Math. Phys., 2019, 0518858] the first complete initial data conditions that guarantee the existence of a Conformal Killing vector on a spacetime evolved by Einstein's equations. The conditions consist of both differential and algebraic equations on the initial data.

M. Ortaggio studied vacuum solutions of Einstein's equations in six dimensions under the assumption that they admit a non-degenerate multiple Weyl aligned null direction and have a "suitable" fall-off of the Weyl tensor at infinity. These solutions were thus completely classified in terms of few parameters and shown to reduce to a family of "generalized" Myers-Perry metrics (including various black hole configurations) [J. High Energy Phys., 2017, 0475200].

In a collaboration with A. Giacomini (Valdivia, Chile), M. Ortaggio also explored a class of ansätze for the construction of exact solutions of the Einstein-nonlinear $\sigma$-model system with an arbitrary cosmological constant in (3+1) dimensions. Thanks to a geometric interplay between the SU(2) field and Killing vectors of the spacetime, the field equations are significantly simplified and explicit solution can be constructed, such as stationary Newman, Unti, Tamburino black holes, uniform black strings, as well as time-dependent solutions such as Robinson-Trautman and Kundt spacetimes, and certain Bianchi IX cosmologies [J. High Energy Phys., 2019, 0508624].

### ***Algebraic topology and geometry, mathematical physics, string field theory, homotopy, gauge fields***

The leading senior researcher M. Markl, has recently extended this group with several postdocs using the funding from the Czech Science Foundation and in particular from the prestigious Praemium Academiae award of the Czech Academy of Sciences. Here we describe his most important results together with the results of U. Schreiber and postdoc J. Vysoký (a member of the team from 2016 to 2018).

*Algebraic topology and geometry*

One important direction of the research was based on the novel notion of an operadic category introduced by M. Markl and M. Batanin (Macquarie University, Australia) in [Adv. Math., 2015, 0447557], initiated by Markl's understanding of the structure of the category of differential graded categories. Operadic categories are today a widely accepted and used concept. M. Markl and M. Batanin consequently developed the notion of Koszulity in operadic categories and, together with Markl's postdoc J. Obradović, proved the Koszulity of some important operads. This resulted in a simple description of their minimal models which, in turn, describe homotopy versions of several objects relevant in geometry, deformation theory, and mathematical physics.

M. Markl, together with A. Lazarev (University of Lancaster, UK) in [Adv. Math., 2015, 0446407] generalized the classical rational homotopy theory of D. Sullivan and D. Quillen to non-connected spaces. As an application, new results on the structure of Maurer-Cartan moduli spaces were obtained. M. Markl made most of the complicated calculations necessary for establishing the model structure on the category of complete Lie algebras.

During the period of the evaluation, there has also been fruitful cooperation of M. Markl with E. Remm (Université de Haute Alsace, France) and V. Dotsenko (Université de Strasbourg, France) on the topics related to the Veronese powers and Koszulity tests for operads. The results will be published soon.

*Mathematical physics, string field theory*

Building on Markl's [J. Noncommut. Geom., 2016, 0460705] where non-symmetric modular operads were introduced, M. Markl together with his former postdoc M. Doubek (Charles University, Prague) characterized in [J. Noncommut. Geom., 2018, 0499147] the operad of diffeomorphism classes of Riemann surfaces with both open and closed boundary components, in the sense of string field theory, as the modular completion of its genus 0 part, modulo the Cardy condition well-known to physicists. This is an ultimate upshot of the approach pioneered in the classical 2001 paper by Markl on loop homotopy algebras [Commun. Math. Phys., 2001, 0175024]. M. Markl did most of the work since M. Doubek tragically died in the course of writing the paper.

Closely related to the algebraic structure of string field theory is [Lett. Math. Phys., 2017, 0476001] by M. Markl and A. A. Voronov (University of Minnesota, Minneapolis, USA) developing a new formalism for the category of IBL-infinity algebras arising in closed string field theory. Markl's contribution was to realize the central role of the convolution product in the algebraic setup which drastically simplified the formalism involved.

B. Jurčo and M. Markl finished the monograph "Algebraic Structure of String Field Theory" written together with I. Sachs (Ludwig-Maximilian-Universität, München, Germany) and M. Doubek. It has been accepted for publication in Springer Lecture Notes in Physics series and will be published in February 2021. The approach of the monograph is based on the formalism of Markl's 2001 paper [Commun. Math. Phys., 2001, 0175024] and the work mentioned above. Because of the tragic demise of M. Doubek in 2016, Markl did most of the writing of the mathematical part. Jurčo shared the work on the physical part with I. Sachs.

B. Jurčo, in close cooperation with the other members of the team, published two papers recognizing the role of strongly homotopy structures in mathematical physics, namely [Fortschr. Phys. – Progress of Physics, 2019, 0518730] written with L. Raspollini (University of Surrey, UK), Ch. **Sämann (Heriot-Watt University, Edinburg, UK) and M. Wolf** (University of Surrey, UK), and [Fortschr. Phys. – Progress of Physics, 2019, 0508773] written with T. Marcelli (University of Surrey, UK) and the coauthors of the previous paper. Jurčo's contribution was his expertise in strongly homotopy Lie algebras gained during his collaboration with M. Markl and other team members. B. Jurčo also supervised J. Vysoký, a former postdoc in the team, and they jointly published [J. Geom. Phys., 2018, 0489051].

*Homotopy and gauge fields*

The Haag-Kastler axioms for algebraic quantum field theory are still the conceptual foundation of any mathematically rigorous approach to quantum field theory. However, despite the original motivation of the Yang-Mills theory, it is well-known that actual gauge theory cannot fit into these axioms since the Haag-Kastler nets provide no room for gauge freedom. U. Schreiber in collaboration with M. Benini and A. Schenkel set out in [Commun. Math. Phys., 2018, 0488852] to rectifying by re-doing the standard analysis of the classical Yang-Mills theory suitable for local quantization, but now with the full gauge theoretic nature reflected in the mathematics, namely by passing to the theory of stacks. Based on this, his co-authors have meanwhile been developing this much further, and the prospect of a gauged "higher geometric" version of the old Haag-Kastler theory appears now to be in reach.

[Adv. Theor. Math. Phys., 2018, 0504774] written by U. Schreiber in collaboration with D. Fiorenza and H. Sato is part of a series of articles where the authors re-analyze the mathematical structure of kappa-symmetric super $p$-brane sigma models through the lens of modern homotopy theory, with the aim to extract information about the elusive mathematical nature of M-theory (more recently reviewed in [arXiv:1903.02834]). The authors derive, for the first time, the mathematical rules for T-duality that had been proposed under the name "topological T-duality" from actual string theoretic data, namely from the structure of the super p-brane supercocycles.

***Homological algebra, commutative algebra, algebraic geometry, and representation theory***

M. Hrbek joined the Institute in July 2017. L. Positselski joined the Institute only in September 2018, and many of his publications, including one of those from 2019, have thus other affiliations. L. Positselski's research in 2015–2019 was focused on the closely related concepts of derived categories of the first and second kind, comodule-contramodule correspondence, and contraherent cosheaves.

*Tilting theory in commutative algebra*

M. Hrbek together with J. Šťovíček (Charles University, Prague) provided a full classification of equivalence classes of tilting modules over any commutative ring in terms of certain finite filtrations of the Zariski spectrum [Forum Math., 2020, 0518738]. This classification provided a basis for the joint paper [Indiana Univ. Math. J., 2020, to appear, see [arXiv:1712.08899] with J. Šťovíček and J. Trlifaj (both Charles University) in which the Zariski locality of tilting modules is established, generalizing the classical result of Raynaud and Gruson.

One of the starting points of [Forum Math., 2020, 0518738] is the observation that cotilting classes over commutative rings are closed under injective envelopes. A naive dualization of this statement would claim that the tilting classes are closed under flat covers. In [J. Pure Appl. Algebra, 2019, 0494437], M. Hrbek showed that this statement holds only for very special types of rings. In particular, it is shown that the class of divisible modules over an integral domain is closed under flat covers if and only if the domain is almost perfect.

*Torsion pairs in derived categories*

The torsion pairs in derived categories, the t-structures, and co-t-structures in particular, provide a framework suited well for both the derived localization theory and the tilting theory of a given ring. Building on the previous results for tilting modules, M. Hrbek established for any commutative ring a bijection between compactly generated t-structures in the derived category and the doubly-infinite Thomason filtrations of the Zariski spectrum [Math. Z., 2019, 0520872]. Together with S. Bazzoni (Università degli Studi di Padova), M. Hrbek proved in the preprint [arXiv:1901.04577] that each definable class in the derived category of an associative ring of weak global dimension at most one is determined on cohomology, generalizing previous results known for hereditary and von Neumann regular rings, and then used this to establish a full classification of t-structures with definable coaisles in the local and commutative case, that is, for valuation domains.

*Weakly curved $A_\infty$ algebras*

The derived categories of modules over dg rings and $A_\infty$ algebras are widely used and well-behaved constructions. When one needs to consider curved $A_\infty$ algebras and modules, as it happens in the Floer-Fukaya theory, the situation becomes much more delicate. There is a vanishing phenomenon, discovered by M. Kontsevich and recorded by L. Positselski in his 2011

AMS Memoir, according to which the theories of curved $A_\infty$ algebras and modules over a field are essentially trivial. A way out of this predicament was worked out by L. Positselski in [Soc. Math. France, 2018, 0505597]: one has to restrict oneself to so-called weakly curved $A_\infty$ algebras over complete topological local rings, which means that the curvature element must be divisible by the maximal ideal of the local ring. Using contramodules over the topological local rings of coefficients as the basic underlying category replacing the vector spaces, a well-behaved formalism of weakly curved $A_\infty$ algebras and modules is developed in [Soc. Math. France, 2018, 0505597]. A construction of semiderived category of $A_\infty$ modules over a weakly curved $A_\infty$ algebra is introduced, and it is shown by examples that this triangulated category can be quite nontrivial, as it should be.

*The tilting-cotilting correspondence*

The tilting-cotilting correspondence is an extension of the comodule-contramodule correspondence to the realm of infinitely generated tilting theory. The $n$-tilting-cotilting correspondence for integer $n$, as constructed by L. Positselski jointly with J. Šťovíček (Charles University) in [Int. Math. Res. Not., 2019, 0520865], is a bijection between complete, cocomplete abelian categories with an $n$-tilting object and an injective cogenerator, on the one hand, and complete, cocomplete abelian categories with an $n$-cotilting object and a projective generator, on the other hand. The *n*-tilting-cotilting correspondence is accompanied by a derived equivalence between the two abelian categories connected by the correspondence. The $n = \infty$ tilting-cotilting correspondence, developed by L. Positselski and J. Šťovíček in [Pac. J. Math., 2019, 0508627], is an infinite homological dimension version of the tilting-cotilting correspondence. An important difference with the finite $n$ version is that one obtains a so-called pseudo-derived equivalence between the two abelian categories involved, instead of a usual derived equivalence.

***Cohomological and categorical methods in differential and algebraic geometry, number theory, and statistics***

*Geometric structures on manifolds and their deformations*

H.V. Lê's joint works with Y-G. Oh (Institute of Basic Sciences, Center for Geometry and Physics in Pohang), L. Vitagliano, A. Tortorella (Università degli Studi di Salerno), and with L. Schwachhöfer (Technische Universität Dortmund) on geometric deformation problems on manifolds with special holonomy, and Jacobi manifolds have been published in [Asian J. Math., 2016, 0460701], [J. Geom. Phys., 2017, 0477507], [J. Symplectic Geom., 2018, 0501808], and in [Osaka J. Math., 2019, 0506686]. H. V. Lê with J. Vanžura also published a paper on the deformation theory of calibrated submanifolds in [J. Geom. Phys., 2017, 0470374], and with D. Fiorenza (University of Rome), L. Schwachhöfer, L. Vitagliano, they proposed a new deformation theory for calibrated submanifolds [arXiv:1804.05732] that has been currently submitted.

*Cohomological invariants of geometric structures and their applications to dynamical systems and variational calculus*

The aforementioned results in the deformation theory of calibrated submanifolds motivated H. V. Lê and her co-authors L. Schwachhöfer (Technische Universität Dortmund) and K. Kawai (Gakushuin University, Tokyo) to write the paper [*Ann. Mat. Pura Appl., 2018,* 0488522] *which contains several important new ideas and results, among them* a new characterization of torsion-free $G_2$-manifolds and Spin(7)-manifolds. These manifolds are central objects in modern Riemannian geometry, M-theory, and Donaldson-Segal's higher-dimensional gauge theory. Their results opened new directions for the study of these manifolds, in particular, they led H. V. Lê and her co-authors to new cohomology invariants of these manifolds published in [*Int. J. Math.* 0497780*], which subsequently motivated her, D. Fiorenza, K. Kawai, and L. Schwachhöfer to* *write a paper on the Sullivan model of smooth manifolds in low dimension [arXiv:1902.08406]. The latter paper has been accepted to Annali della Scuola Normale Superiore di Pisa.* H. V. Lê also collaborates with J. F. Barraud (Université Toulouse III – Paul Sabatier), A. Gadbled (Uppsala University) and R. Golovko (Charles University) on a paper on Novikov fundamental group and their application in variation calculus [arXiv:1710.10353], which has been accepted to IMRN (International Mathematical Research Notices).

*Geometric and categorical methods in statistics*

H. V. Lê's joint work with N. Ay and J. Jost from the Max-Planck-Institute in Leipzig and with L. Schwachhöfer on Information Geometry – a new domain devoted to geometric methods in mathematical statistics – has been published in several papers and a book. In their paper [*Probab. Theory Relat. Fields, 2015,* 0444389]*, they introduced notions of parametrized measure models and tensor fields on them that exhibit the right behaviour under statistical transformations. In particular, they proved a full generalization of the original result of Chentsov in 1972 to infinite sample sizes. The theory of parameterized measure models has been further developed in their paper [Bernoulli, 2018,* 0486135] *and culminated in their book Information Geometry*, published in the series *Ergebnisse der Mathematik und ihrer Grenzgebiete* of the Springer. Before their work [Probab. Theory Relat. Fields, 2015, 0444389] people used only differential geometric techniques to study statistical models which are smooth manifolds, and most techniques were applied only to finite-dimensional statistical models. In this book, they extend these techniques to statistical models that can be parameterized by a smooth Banach manifold. Then they show many applications of their approach, e.g. a general version of the classical Cramer-Rao inequality. In her paper [*Ann. Inst. Stat. Math., 2017,* 0476158]*,* H. V. Lê proposed a new generalization of the Chentsov theorem, and in a recent paper [arXiv:1912.02090], she proposed a new geometric theory of statistical models that encompasses all existing statistical models. This paper has been published in 2020.

*Vertex algebras, their characters on Riemann surfaces and applications in number theory, differential and algebraic geometry*

*The Prime form and the Szegő kernel are among the main tools in computations of characters for vertex operator algebras in conformal field theory defined on Riemann surfaces. In* [Adv. Theor. Math. Phys.*,* 2018, 0505713] *A. Zuevsky used a geometric representation for the Szegő kernel and derived relations for corresponding prime forms on genus* $g+1$ *and genus* $g$ *Riemann surfaces. In the analytical number theory, it is important to establish connections between algebraic structures and complex analysis. In [*Internat. J. Theoret. Phys., 2016, 0448470*] it is shown that the Hardy spaces for two doubles of Riemann surfaces are isometrically isomorphic. In [*Internat. J. Modern Phys. B, 2017, 0465183*] the notion of vertex operator cluster algebras over a noncommutative set of variables arising from conformal field theory was introduced. Starting from character theory for vertex algebras, Zuevsky has computed in [*Int. J. Number Theory, 2018, 0491679]*, [*Šiauliai Math. Semin. 2016, 0475230] *new formulas for complex powers of the classical eta-function.*

## Research activity and characterisation of the main scientific results

### *Boundary value problems for ordinary differential equations*

This topic constitutes the core of the team's research (members involved: R. Hakl, A. Lomtatidze, S. Mukhigulashvili, A. Rontó, J. Šremr, M. Tvrdý). Ideas and principles earlier formulated by the team members allowed them to develop new methods and techniques to tackle challenging complex problems that had long remained unsolved. A significant attention was paid to the periodic boundary value problem for second-order non-autonomous nonlinear differential equations in both regular and singular cases. This question is also important due to its close relation to the stability analysis.

The extensive work [Mem. Differ. Equ. Math. Phys., 2016, 0460328] by A. Lomtatidze is devoted to the above-mentioned periodic problem and provides comprehensive and detailed information about this important and classical subject. Many new facts are revealed and systematically described, their interplay with other results is outlined. The methodology used is original and innovative. It is based on the in-depth analysis of numerous subtle questions of the qualitative theory, in which the author succeeded to obtain very strong results which substantially extend the previously known ones. In particular, the technique of differential inequalities for linear equations with sign-changing coefficients is developed. Necessary and sufficient conditions for the validity of the so-called maximum and anti-maximum principles are established, effective sufficient conditions are presented, and explicit lower bounds for positive solutions are established. Based on the theory developed for the linear case, conditions guaranteeing the solvability of the periodic boundary value problem for nonlinear equations are established both for regular equations and for those with a singularity in the phase variable. The resonant case, where the linear part has a nontrivial solution, is also considered. It is worth mentioning that, in the currently available related works, this important issue is usually addressed using fixed point theorems in cones. The results obtained here by A. Lomtatidze are based on much more sophisticated techniques of differential inequalities and *a priori* estimates. As a consequence, his results are sharper than those obtained by the fixed point techniques, e.g. in the case where the phase singular term is not autonomous or the inhomogeneous (absolute) term of the equation changes its sign. Not only equations with the phase singularity at the origin but even those with infinite growth in a neighbourhood of plus infinity are covered by these results. For a wide enough class of equations with the linear part at resonance, the conditions are necessary and sufficient. Moreover, in contrast to the previously known results, the space singular term is not assumed to have a power-growth character. A complementary part of the well-known Lazer-Solimini conditions is established and an open problem on a Fredholm-like alternative for phase singular equations stated by R. Manásevich in 1992 is partly solved. As for the validity of the anti-maximum principle for the linear case, the assumption of the negativity of the coefficient, which is usually present in the existing literature, is not necessary here. The above-mentioned results were further developed in the papers [Georgian Math. J., 2017, 0475573], [Nonlinear Anal., Real World Appl., 2018, 0480063], [Electron. J. Differ. Equ., 2018, 0486017], and [Georgian Math. J., 2016, 0466803]), where the class of equations under consideration was extended. The importance and novelty of the obtained results can be exemplified by Duffing type equations with nonconstant coefficients, which are useful in mathematical models of various oscillators. The questions on the existence and multiplicity of periodic solutions to the autonomous Duffing equations have attracted much attention and many interesting results are currently known. A number of existence results are also available for the non-autonomous case on the common assumption that the non-linear term is sub-linear. However, the Duffing-type equations are essentially super-linear and, therefore, they had long remained practically unstudied due to absolute impossibility to apply standard methods. New techniques developed in the works of A. Lomtatidze and J. Šremr allowed them to handle this challenging problem. Moreover, in the autonomous case, the results obtained in [Nonlinear Anal., Real World Appl., 2018, 0480063] coincide with the results in the classical case (e.g. multiplicity results for the existence of sign-constant solutions and oscillatory solutions), which justifies their optimality.

An investigation complementary to the above-described research was conducted by R. Hakl. He focused on the differential equations with singularities in the phase variable in the absence of a linear part (this can be treated as a resonant case with zero linear part), and the research was based on an essential extension of methods developed in his previous works with M. Zamora and P. J. Torres. Differential equations of such kind can be divided into three basic classes according to the behaviour of the singular term as the phase variable approaches the origin: equations with repulsive, attractive, and indefinite or attractive-repulsive type singularity. While the repulsive case is the subject of numerous works and nowadays there is a quite wide theory on the topic, the attractive case is much more difficult due to serious obstacles in the application of standard tools. That is why in most of the papers devoted to the attractive case, the equations were studied under certain regularity conditions on the coefficients (usually they are supposed to be continuous or bounded), and the singularity term is assumed to be either autonomous or bounded from below by some positive constant. It is worth mentioning here that both restrictions (regularity conditions and autonomous singular term) are essential for the classical methods to handle the problem with singularities. R. Hakl in collaboration with M. Zamora from the University of Oviedo succeeded to overcome this fundamental difficulty and studied the existence of a periodic solution to the second-order ordinary differential equation with an attractive singularity in the phase variable in [Ann. Mat. Pura Appl., 2016, 0459380] without the both mentioned restrictions. They have developed a sophisticated technique allowing to investigate the periodic problem in the attractive case in a more general setting than ever before. They established theorems revealing a very interesting fact that explained why the standard methods failed – the existence of a periodic solution is closely related to the order of singularity, the order of zeros of the non-autonomous singular term, and the regularity of the forcing term.

The experience gained during the study of the attractive case turned up to be very useful for the investigation of equations with singularities of indefinite type. Thus in [J. Differ. Equations, 2017, 0474090] R. Hakl and M. Zamora studied a generalized Emden-Fowler equation with a negative exponent. Periodic solutions to equations of such type appear in various physical contexts where the weight function depends closely on the nature of every particular model. The periodic problem for the Emden-Fowler type equation with a phase singularity, in spite of its simple looking structure, is a challenging problem due to the lack of any *a priori* estimate over the set of possible periodic solutions. Such *a priori* estimates, in general, are essential to apply any tool of functional analysis guaranteeing the existence of a solution. That is why the problem had been studied very little so far, and under very restrictive conditions imposed on the weight function (piecewise constant or sufficiently smooth with simple zeros). To overcome these difficulties, R. Hakl and M. Zamora proposed a completely new strategy of proving a continuation theorem on an unbounded open set where the requirements on *a priori* estimates are weakened. Using this new strategy together with delicate estimates, they established an existence theorem for the periodic problem in question for a quite general class of weight functions. Again, the established theorem revealed the reason for the inapplicability of the standard methods – the existence of a periodic solution is closely related to the order of singularity and the order of zeros of the weight function.

An important topic that attracts much attention among physicists and mathematicians – the Kepler problem on a sphere – leads to a differential equation with two singularities of indefinite type. In [Adv. Nonlinear Stud., 2019, 0504396], R. Hakl and M. Zamora succeeded to obtain a natural generalization of their previous results to equations with two singularities (see also [Can. J. Math., 2018, 0484742]). Their techniques turned out to be fruitful also in this situation. In contrast to the case of a single singularity, the multiplicity of solutions appeared as a new phenomenon. For a quite wide class of weight functions R. Hakl and M. Zamora established conditions guaranteeing the existence and multiplicity or nonexistence of periodic solutions. More precisely, the periodic problem has at least two solutions provided the weight function is sufficiently small, and there is no solution provided the weight function is sufficiently large. It is worth mentioning that this problem had also been studied insufficiently in the literature due to the same analytical complications that are present for equations with one singularity of indefinite type.

An interesting effect arising in studies of the circulation of a fluid in certain mechanical systems is called the Liebau or valveless pumping phenomenon; this topic has been recently addressed by M. Tvrdý. The term "valveless pumping" refers to the conveyance of liquid fluids in mechanical systems that have no valves to ensure the preferential direction of flow. Such phenomena appear e.g. in the models of blood circulation in the cardiovascular system, in some other models from microfluidics, or, at large scales, in oceanic currents. The model is a non-linear ordinary second-order differential equation with a singularity containing the second power of the derivative of the unknown function. First results about the existence and stability of periodic solutions have been delivered already in 2014 by M. Tvrdý together with J. A. Cid and G. Propst. In the period 2015–2019, M. Tvrdý in cooperation with J. A. Cid, G. Infante and M. Zima presented in [J. Math. Anal. Appl., 2015, 0436523] and [Nonlinear Anal., Real World Appl., 2017, 0469633] essentially improved results for a nonlinear problem containing the Liebau model as a special case. In particular, for the first time, the case of sign-changing external periodic forces had been covered.

The study of periodic solutions of ordinary differential equations is closely related to properties of the corresponding translation operator. Related questions for problems more general than the periodic one have recently been studied by A. Rontó. In the work [Bull. Sci. Math., 2016, 0457324] carried out together with M. Fečkan and N. Dilna, the use of a suitable extension of the notion of translation operator allowed to investigate an interesting class of systems of weakly non-linear ordinary differential equations admitting solutions which possess a symmetry property generalizing, e.g., periodicity, antiperiodicity and affine periodicity. One may note that the study of affine-periodic solutions currently attracts much attention. This work also significantly extends a number of previous results on antiperiodic solutions of nonlinear equations. Effective conditions ensuring the existence of a locally unique symmetric solution were established both in non-resonance and resonance cases (when there are constant functions having the property in question). The important question on the Lyapunov stability of such solutions was studied in detail. Two kinds of theorems on asymptotic stability were proved, which provide sufficient conditions with logarithmic norms and, respectively, effective spectral conditions involving directly the spectral abscissa of a linearization matrix at the generating solution. It should be noted that these explicit results are specific for the class of equations under consideration and do not follow from the existing theorems on equivariant differential systems. The conditions are, in a certain sense, optimal. The proofs include specially developed new techniques extending the apparatus for the investigation of other related problems (e.g., generalized logarithmic norms, new Krasnoselskii-type lemmas on nested neighbourhoods). The theorems obtained here correlate well with the classical results and imply, in particular, a certain analogue of the Bogolyubov Second Theorem for a non-periodic case.

The rest of the works belonging to this section deal with other widely studied types of boundary value problems; in particular, the focal, Dirichlet, and Dirichlet-Neumann boundary conditions were investigated. In addition to newly introduced techniques adapted to particular classes of problems, this research also benefits from the previous theoretical results of the team.

Two-point boundary value problem (with boundary conditions of the focal point type) for linear ordinary differential equations of the fourth order is studied in [Georgian Math. J., 2017, 0475575] by S. Mukhigulashvili together with M. Manjikashvili from the Tbilisi State University. Along with the independent theoretical interest in this topic, an analysis of the linear case is necessary for a thorough investigation of focal problems for fourth-order nonlinear equations. The linear problems are important for the study of the spectral properties of the fourth-order linear operators or for the study of the elastic beam equations. This kind of boundary value problem was previously studied mainly for autonomous equations or in the so-called half-autonomous case. In addition, rather little is known about the validity of the maximum principle. In comparison with previously known results, this work describes several important steps forward: some efficient integral conditions are established guaranteeing that the space of nontrivial solutions to the corresponding homogeneous problem is at most one-dimensional (and therefore, the resonant case is not excluded for the inhomogeneous problem). Furthermore, the authors establish conditions ensuring that the number of zeros of any solution to the homogeneous boundary value problem is less than a given natural number; results of this type are very important for studying the spectrum of the fourth-order linear operator. Various integral conditions are established guaranteeing the unique solvability of the inhomogeneous problem, even in the case when the corresponding homogeneous part is at resonance. The proofs of the main results involve a number of sophisticated estimates based on the techniques of mathematical analysis. The investigation of the fourth-order equations is a challenging task facing significant complications due to the absence of any analogues of the Sturm separation and comparison theorems. The estimation of the number of zeros of solutions is therefore an extremely difficult problem in this case.

In contrast to the Dirichlet problem, the mixed problem for second-order equations is still studied very poorly in the literature. The work [Math. Nachr., 2017, 0471008] by S. Mukhigulashvili is devoted to the solvability of mixed boundary value problems (Dirichlet-Neumann type condition) for nonlinear second-order ordinary differential equations at resonance. It is worth mentioning here that also the Dirichlet problem at resonance is studied mainly in the autonomous case, i.e. in the case where the linear part of the corresponding nonlinear equation has constant coefficients. Here, S. Mukhigulashvili succeeded to study the non-autonomous case and to obtain an analogue of the so-called Landesman-Lazer condition (well known for the periodic problem) guaranteeing the solvability of the problem at resonance. The technique used to establish his results is based on delicate estimates in a combination with suitable *a priori* estimates.

The study of the Dirichlet problem in a natural way includes the question on its well-posedness. A. Lomtatidze together with Z. Opluštil from Brno University of Technology established conditions guaranteeing the well-posedness of the Dirichlet problem for the second-order linear differential equations where the input functions are locally Lebesgue integrable and may have singularities at the endpoints of the interval. They also introduced examples that prove the optimality of the conditions (see [Georgian Math. J., 2015, 0448472]).

### *Boundary value problems for functional differential equations*

The interesting topic dealing with boundary problems for functional differential equations, which is naturally related to the corresponding theory for ordinary equations, forms an independent section in the research of the team (members involved: R. Hakl, S. Mukhigulashvili, A. Rontó). As the mathematical models in applied sciences are becoming more realistic, the influence of delay (and, more generally, argument deviation) is naturally taken into account, and one is increasingly faced with the necessity to formulate models in this more complicated class of equations. This is a traditional research direction of the team, with numerous results published in the past, including one monograph. A number of new results have also been obtained in the last five years.

As a significant achievement one can mention the paper [Topol. Methods Nonlinear Anal., 2019, 0521050] by S. Mukhigulashvili and V. Novotná from Brno University of Technology. The Dirichlet and mixed (Dirichlet-Neumann) problems for both linear and nonlinear second-order integro-differential equations were studied and optimal conditions guaranteeing the existence and uniqueness of a solution were established. The class of integro-differential equations already represents all peculiar features and various analytical difficulties of the broader class of functional differential equations. The theorems obtained can be applied to wide enough classes of boundary value problems for both ordinary and functional higher-order differential equations. This kind of results is motivated, in particular, by the possibility to reduce higher-order equations to equations of this type using (explicitly given) Green functions of certain auxiliary problems.

R. Hakl in collaboration with A. Domoshnitsky and B. Půža studied the multi-point boundary value problems for linear functional differential equations and found effective solvability conditions in [Georgian Math. J., 2017, 0475574]. These results, which cover the frequently studied classical cases (e.g., initial, periodic), also allowed to investigate other less common types of boundary conditions. The main theorems were proved by using original techniques based on the authors' earlier results on functional differential inequalities.

The underlying ideas were successfully used in other interesting problems concerning evolution equations with discontinuous trajectories, equations with symmetries, and strongly singular equations of order greater than two. The existence of almost periodic solutions for semi-linear abstract parabolic evolution equations with impulses at state-dependent time instants was studied by R. Hakl, M. Pinto, V. Tkachenko, S. Trofimchuk [J. Math. Anal. Appl., 2017, 0464478]. They suggested weaker conditions ensuring the absence of an undesirable and difficult to handle effect – the so-called beating phenomenon. Motivated by previous studies on related classes of ordinary equations, A. Rontó in collaboration with N. Dilna and M. Fečkan [Symmetry, 2019, 0521842] studied a rather general functional differential equation and obtained conditions ensuring the existence and uniqueness of a solution possessing an algebraic property generalizing periodicity. In this situation, the inapplicability of methods employed earlier in the ordinary case was overcome by using the framework of boundary value problems for functional differential equations. Strongly singular higher-order nonlinear functional differential equations were studied by S. Mukhigulashvili in [Ital. J. Pure Appl. Math., 2015, 0457320] and an important *a priori* boundedness principle was established for rather wide classes of boundary value problems. The analytic techniques developed by the author allowed him to effectively deal with general nonlocal boundary conditions. The proof of the *a priori* boundedness principle is based on new Agarwal-Kiguradze-type theorems guaranteeing the Fredholm property of strongly singular higher-order linear equations with argument deviations under the corresponding boundary conditions. Effective sufficient conditions guaranteeing the solvability of the two-point right-focal problems for higher-order equations were found by S. Mukhigulashvili and B. Půža [Bound. Value Probl., 2016, 0446810]). These problems are singular in the sense that the right-hand side term of equation may have non-integrable singularities at the beginning of the interval, which complicates their investigation and requires the development of adequate tools. Similar difficulties arise when the behaviour of the equation is observed on an unbounded interval. It turned out that the methods developed by the team are fruitful in such cases as well. In this way, in the case of unbounded interval, R. Hakl together with M. Aguerrea from Catholic University of Maule succeeded to establish effective conditions sufficient for the existence of a global solution for the first-order nonlinear functional differential equation with Volterra-type operators [Proc. R. Soc. Edinb., Sect. A, Math., 2017, 0484751]. Criteria under which the global solution is bounded and non-negative in a neighbourhood of minus infinity were found and new results on the existence of solutions positive on the whole real line were obtained. The theorems were applied to certain models arising in natural sciences, where the ultimate positivity of the observed quantity is a very natural requirement. The question on bounded solutions of systems of nonlinear functional differential equations was addressed by R. Hakl and J. Vacková in [Funct. Differ. Equ., 2017, 0499722], where new existence results for bounded solutions are obtained using *a priori* estimates and previous developments concerning functional differential inequalities on bounded intervals, which are applied in this case through suitable truncation techniques and passing to the limit. The importance of these results is in the considerably general setting given by the fact that the operator in the equation is not assumed to possess the Volterra property. Some of the works mentioned in this section deal with matters that are, in a sense, bordering on the asymptotic theory (equations with non-integrable singularities in the time variable, behaviour of solutions at infinity, boundedness of solutions).

### *Asymptotic theory*

Certain issues arising in the qualitative analysis of boundary value problems essentially overlap towards asymptotic theory (e.g., the positivity of Green's function, which is one of the major important properties studied in boundary value problems, is, in fact, very closely related to non-oscillation). The asymptotic theory has been among the long-term research directions of the team (members involved: A. Lomtatidze, P. Řehák, J. Šremr); numerous papers and a book were published on this subject. The results obtained deal with asymptotic properties of solutions of differential and difference equations. They concern, more specifically, the study of oscillatory behaviour of solutions and asymptotic formulas for their qualitative representation.

Topics related to asymptotic formulas were studied mainly by P. Řehák. The principal idea behind was the systematic use of the theory of regular variation. This, in a combination with other relevant tools (such as the Riccati technique, comparison theorems, reciprocity principle, certain transformations of independent and dependent variables, and principal solutions), has led to gaining a rather precise information about asymptotic behaviour. Mainly the half-linear equations were treated; this is a complicated class of equations, the systematic study of which dates back to the original works of Emden and Fowler. Asymptotic relations describing the growth of solutions of half-linear differential equations at infinity are established in the works of P. Řehák [Acta Math. Hung., 2015, 0446870], [Electron. J. Differ. Equ., 2016, 0463600], [Appl. Math. Comput., 2017, 0461884] and [Differ. Integral Equ., 2016, 0461877]. Many of the observations presented there are new even in the linear case. The use of the Riccati technique, the classical result for differential inequalities, and various estimates, including the Hölder inequality, allowed the author to get asymptotic estimates for the positive increasing solution (see [Acta Math. Hung., 2015, 0446870]). In [Differ. Integral Equ., 2016, 0461877] it is shown that the increasing, resp. decreasing solutions belong to the de Haan class $\Gamma$, resp. $\Gamma_{-}$ under suitable assumptions. Asymptotic formulas are derived also for slowly varying functions. The study continued in [Electron. J. Differ. Equ., 2016, 0463600] and [Appl. Math. Comput., 2017, 0461884], where further methods were proposed. In particular, in [Appl. Math. Comput., 2017, 0461884], the asymptotic linearization device was used to treat more general settings and also, in some sense, critical and difficult settings. The results obtained are optimal in the sense that, for the linear case, they coincide with already known results (if any had earlier existed). Along with the differential equations, the asymptotic behaviour of solutions to non-oscillatory linear difference equations was also studied. Conditions in terms of regular variation, which guarantee that all the eventually positive solutions satisfy certain asymptotic formulas, were found in [J. Difference Equ. Appl., 2016, 0458376]; the discrete regular variation played an important role in the analysis. Properties of solutions to the $n$-dimensional systems of differential equations were also investigated. The particular case of higher-order equations was treated in [Math. Slovaca, 2016, 0470375] in the differential and $q$-difference setting. For higher-order linear equations, conditions guaranteeing the existence of generalized regularly varying solutions were found. The proofs are based mainly on the classical Poincaré and Perron theorems and on certain transformations.

More general nonlinear systems of first order differential equations of a cyclic type were studied by P. Řehák and S. Matucci in [Math. Nachr., 2015, 0445822]. The attention was focused on difficult classes of solutions, such as strongly increasing and strongly decreasing solutions. Conditions were established under which such solutions exist and all of them are regularly varying functions, and asymptotic representations were obtained. The asymptotic results obtained were applied in several important cases including higher-order nonlinear differential equations, equations with a generalized $\varphi$-Laplacian, and nonlinear partial differential systems.

One of the well-known oscillation results for linear second-order ordinary differential equations is the Kamenev criterion, which requires that the upper bound of a certain expression is equal to plus infinity. In the literature, numerous generalizations of this criterion can be found for various differential equations and their systems. However, there are only few oscillation results complementing the Kamenev criterion, i.e., results which are applicable in the case when the upper bound of the Kamenev type expression is finite or minus infinity. A. Lomtatidze and J. Šremr, together with M. Dosoudilová, obtained a new Kamenev-type oscillation criterion and its counterpart for the two-dimensional Emden-Fowler type differential system with locally integrable input functions $p$ and $g$ [Nonlinear Anal., Theory Methods Appl., 2015, 0443123], where the function $g$ is supposed to be positive. It is essential that the theorems cover both cases according to whether the integral of $g$ over the real half-line is finite or not. In particular, these results have allowed the authors to establish a new Hartman-Wintner-type theorem. The latter is particularly interesting because, for integrable $g$, the assumption on the positivity of $p$ is not needed, while it was essentially required in all the earlier works.

Two-dimensional systems of the Emden-Fowler type include, in a natural way, the half-linear second-order differential equations. Such equations were studied along these lines by J. Šremr in the work [Electron. J. Qual. Theory Differ. Equ., 2016, 0461454] presenting interesting oscillation and conjugacy criteria on the entire real line. The theorems extend and complement results currently available in the existing literature; they are new even for the linear second-order differential equations. Oscillatory properties of second-order equations with delay were treated by J. Šremr in collaboration with Z. Opluštil. Their paper [Monatsh. Math., 2015, 0446759] contains new Myshkis-type oscillation criteria, which generalize and, under some natural additional assumptions, improve previously known results.

***Constructive methods for boundary value problems***

It is natural to complement the study of solvability of boundary value problems and the qualitative analysis of solutions with description of the way, how the solutions can be approximately constructed. The latter task, with a very few exceptions related to monotone iterations, is commonly treated by methods totally different from those used in the solvability analysis, so that the existence of the solution in question is either assumed beforehand or proved using different tools. The two questions can be, in a sense, combined together by using parameterization-type methods. This is a topic belonging to the team’s scope (member involved: A. Rontó). First-order systems of nonlinear ordinary differential equations were considered in this context; some previously obtained results were extended and new problems and techniques were studied. In [Appl. Math. Comput., 2015, 0436528], [Electron. J. Qual. Theory Differ. Equ., 2016, 0464474], the authors showed that parameterization-based techniques suggested earlier for two-point problems were shown to be applicable, after suitable modifications, to the problem with a general non-local boundary condition. The works [Appl. Math. Comput., 2016, 0452566], [Georgian Math. J., 2017, 0475572] deal with systems of ordinary differential equations with state-dependent jumps (i.e., with discontinuities of the first kind occurring at the instants of intersection of the trajectory with a given surface; these time instants are thus not fixed but depend on the values of the solutions). Parametrization techniques were developed allowing one to effectively construct approximate solutions of two-point boundary value problems for such systems. Note that almost no effective methods are currently available for this kind of problems, and the approach mentioned, which fills the gap in a certain way, is rather natural in this context.

The ideas based on parametrization can be successfully used in other situations; several interesting problems were recently addressed by A. Rontó in collaboration with B. Půža, M. Rontó, and N. Shchobak. In [Miskolc Math. Notes, 2017, 0475998] it was shown how a suitable parametrization helps one to approximately solve the problem on solutions of non-linear ordinary differential equations vanishing at a certain *a priori* unknown point. The work [Ukr. Math. J., 2017, 0492746] deals with the system of nonlinear first-order ordinary differential equations under two-point boundary conditions and focuses on solutions every component of which vanishes once on the given interval, while the points where they vanish are to be determined together with the solution.

For a practical application, it is natural to combine parametrization methods with other suitable techniques. The work [Electron. J. Qual. Theory Differ. Equ., 2018, 0491056] describes and justifies a version of the method from [Appl. Math. Comput., 2015, 0436528] where the polynomial interpolation over the Chebyshev nodes is used. This version is suitable for ordinary differential systems satisfying additional regularity conditions in the time variable and has the advantage of significantly easier construction of approximations (in particular, much faster computation using suitable software). The practical use of the approach had been tested on various concrete examples using computations in Maple, which confirmed the general experience (good quality of approximation within a relatively little number of steps).

### *Integration theory and generalized differential equations*

M. Tvrdý follows the tradition of the former Department of Ordinary Differential Equations and Integration Theory, established by Jaroslav Kurzweil who was the founder of the concept of the gauge integral, nowadays widely investigated and worldwide accepted as very powerful and useful. Throughout the period 2015–2019 a detailed study of the properties of the Kurzweil-Stieltjes integral has been done. This research was summarized in the monograph [World Scientific, 2019, 0494242] written by M. Tvrdý in collaboration with his former student and the former member of the team G. A. Monteiro and A. Slavík from the Charles University. The book is devoted to the development of the theory of the Kurzweil-Stieltjes integral and its applications. It partly summarizes and completes results that have already been known, or have been obtained and published by the authors before. However, the major part of the results are new, e.g. those concerning the relationship between the Kurzweil-Stieltjes- and other Stieltjes-types integrals, the classes of functions adjoint with respect to the integral, generalized elementary functions, variations-of-constant formula for generalized linear differential equations in the space of regulated functions, functionals on the spaces of regulated functions, and integration on time scales and applications to time scale dynamic systems. The monograph was awarded the 2019 Annual Prize of the Dean of the Faculty of Mathematics and Physics of Charles University.

In the theories of Lebesgue integration and of ordinary differential equations, the Lebesgue Dominated Convergence Theorem provides one of the most widely used tools. An available analogue in the Riemann or Riemann-Stieltjes integration is the Bounded Convergence Theorem, sometimes called also the Arzelà or Arzelà-Osgood or Osgood Theorem. In the setting of the Kurzweil-Stieltjes integral for real valued functions, its proof can be obtained by a slight modification of the proof given for the $\sigma$-Young-Stieltjes integral by T. H. Hildebrandt in his monograph from 1963. However, it is clear that Hildebrandt's proof cannot be extended to the case of Banach space valued functions. Moreover, it essentially utilizes the Arzelà Lemma that does not fit too much into elementary text-books. In [Monatsh. Math., 2016, 0460232], the proof of the Bounded Convergence Theorem for the abstract Kurzweil-Stieltjes integral in as elementary setting as possible has been given. Finally, rather unexpectedly it turned out that the theory of the Kurzweil-Stieltjes integral can be applied, together with the hysteresis theory, to models describing the behaviour of financial markets [Math. Bohem., 2016, 0459263] and other related papers mentioned in the report of the EDE team.

## Research activity and characterisation of the main scientific results

### *Mathematical analysis of fluid dynamics*

Problems in fluid dynamics constitute a dominant part of the team activity in both respects: The number of researchers involved in the topic and the scientific production.
It received substantial support in 2012 when E. Feireisl obtained his ERC Advanced grant. This sub-team consists of H. Al Baba, T. Bodnár, M. Caggio, P. Dell'Oro, E. Feireisl, A. Ghosh, R. Hošek, S. Kračmar, O. Kreml, P. Kučera, V. Mácha, M. Michálek, H. Mizerová, T. Nakatsuka, Y. Namlyeyeva, Š. Nečasová, J. Neustupa, H. Petzeltová, A. Radošević, J. Scherz, S. Schwarzacher, Z. Skalák and I. Straškraba.

*Compressible fluids*

A method based on the concept of relative energy and weak convergence has been proposed to attack problems of multiscale character, in particular in meteorology. E. Feireisl in collaboration with R. Klein (FU Berlin) – leading expert in the field – applied the method to problems of atmospheric stratification. The result is also based on careful analysis of acoustic waves [Math. Models Methods Appl. Sci., 2016, 0456564]. Similar techniques have been developed to study problems of fluid motion in the presence of a magnetic field, notably the models of plasma [Math. Models Methods Appl. Sci., 2015, 0435997]. Last but not least, the hydrodynamic models of liquid crystals have been investigated [Ann. Mat. Pura Appl., 2015, 0446507]. Further results concerning models involving random forces, notably the Cahn-Hilliard-Navier-Stokes system have been obtained in collaboration with M. Petcu (Université de Poitiers) [J. Differ. Equations, 2019, 0504395].

E. Feireisl and his collaborators also extended previous results in singular limits, homogenization, and relative entropy inequality. The method is explained in the extended second edition of the monograph Singular Limits in Thermodynamics of Viscous Fluids [Birkhäuser, 2017, 0483705]. The novelty in the extended theory of singular limits for problems with low regularity of solutions consists in the concept of generalized measure-valued or dissipative solutions. The relative energy method has been adapted to accommodate the so-called Reynolds defect that makes it applicable to problems like compressible Euler system. The results include the stability of the primitive system even in a large class of generalized solutions as long as the target problem in the singular limit process admits a smooth solution. The method has been also used to prove uniqueness of certain non-smooth solutions to the Euler system admitting one-sided Lipschitz estimates on the gradient. This is the case, for example, of rarefaction waves and their perturbations. These results have been achieved in collaboration with several foreign research institutions, notably (J. Březina, Kyushu University, S. Ghoshal, Tata Institute, Bangalore, A. Novotný, Université de Toulon) [Commun. Partial Differ. Equations, 2019, 0508339], [J. Math. Soc. Japan, 2018, 0495259], [SIAM J. Math. Anal., 2016, 0466755]. Further, book [Birkhäuser/Springer, 2016, 0466780] of E. Feireisl, T. Karper from the University of Trondheim, and M. Pokorný from the Charles University offers an essential introduction to the mathematical theory of compressible viscous fluids. Its main goal is to present analytical methods from the perspective of their numerical applications.

The relative entropy inequality was used as a fundamental tool in [Czech. Math. J., 2019, 0506884] for proving the weak-strong uniqueness property and also in [J. Math. Fluid Mech., 2017, 0480573] for proving that the one-dimensional Navier-Stokes-Fourier system can be rigorously derived as a limit of the full 3D case when two dimensions shrink to zero.

Singular limits in systems with radiation have been studied by Š. Nečasová in cooperation with B. Ducomet (Université Paris-Est) and further collaborators. The results include singular limits in the low Mach number regime [Discrete Contin. Dyn. Syst., 2018, 0489042], inviscid limits [Nonlinear Anal., Theory Methods Appl., 2017, 0476961], dimensional reduction in astrophysics [Asymptotic Anal., 2018, 0493687], [J. Math. Fluid Mech., 2018, 0489961], diffusion limit in the full system with radiation and nonrelativistic limits [J. Math. Fluid Mech., 2015, 0443967], [Analysis, 2015, 0444057], [J. Hyperbolic Differ. Equ., 2016, 0461567]. Again, the relative entropy inequality plays a substantial role here.

Š. Nečasová studied together with B. Ducomet (Université Paris-Est) and X. Blanc (Université Paris Diderot) the existence of classical solution of full inviscid system coupled with electromagnetic field with or without a damping term, see [Adv. Nonlinear Anal., 2019, 0502141; Topol. Methods Nonlinear Anal., 2018, 0494449]. Moreover, together with B. Ducomet she showed the existence of the classical solution of Cauchy problem for a full system with radiation, see [Ann. Sc. Norm. Super. Pisa, Cl. Sci., 2015, 0443535]. Together with B. Ducomet and her PhD student M. Kobera she investigated the coupling of the full system with radiation and magnetic field and proved the existence of weak solutions, see [Acta Appl. Math., 2017, 0475999].

The compression effects in heterogeneous media with maximal packing constraint was studied by Š. Nečasová together with D. Bresch (Université Savoie Mont Blanc) and C. Perrin (Université de Marseille). The limit in terms of singular pressure and singular bulk viscosity has been identified in [J. Éc. Polytech. Math., 2019, 0505707].

E. Feireisl and his collaborators proposed original applications of a new theory in inviscid dynamics, namely the convex integration introduced by De Lellis and Székelyhidi. It was shown that adding the energy equation as an integral part of the complete Euler system does not solve the problem of ill-posedness even in the class of entropy admissible solutions. Examples of non-uniqueness have been obtained also in the case of stochastic forcing [Anal. PDE, 2020, 0523860]. Another important contributions to the theory of inviscid compressible fluids were made by M. Michálek, a former PhD student of E. Feireisl [Arch. Ration. Mech. Anal., 2018, 0490764], [Commun. Math. Phys., 2017, 0474807].

As an extension of the De Lellis-Székelyhidi theory of non-uniqueness of solutions to the incompressible Euler system, questions related to uniqueness and non-uniqueness of various types of solutions to hyperbolic conservation laws were intensively studied by the young researcher O. Kreml. He started working on this new promising subject together with C. De Lellis and E. Chiodaroli during his stay in Zürich in 2013–2014. He focused on the problem of non-uniqueness of admissible weak solutions to the isentropic Euler system in various settings of initial data (Lipschitz, Riemann), [Commun. Pure Appl. Math., 2015, 0444193; Nonlinearity, 2018, 0488708; Electron. J. Differ. Equ., 2018, 0489412]. Similar questions were addressed also for the Euler-Fourier system with the participation of E. Chiodaroli, C. De Lellis, E. Feireisl, and J. Březina. This was also the main topic of O. Kreml's project GJ17-01694Y supported by the Czech Science Foundation. Uniqueness of 1D rarefaction waves was proved for both isentropic Euler and full Euler systems in collaboration with E. Feireisl and A. Vasseur (University of Texas in Austin), [J. Hyperbolic Differ. Equ., 2015, 0448120; SIAM J. Math. Anal., 2015, 0445718].

Measure-valued solutions to isentropic Euler system and their properties were studied in a joint work with E. Chiodaroli, E. Feireisl and E. Wiedemann (Universität Ulm) and generalized measure-valued solutions to general systems of hyperbolic conservation laws were studied in a joint work with P. Gwiazda and A. Świerczewska-Gwiazda (both University of Warsaw), [Ann. Mat. Pura Appl., 2017, 0476952; Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 2020, 0524147].

Stochastic effects in compressible fluids were studied by E. Feireisl and his collaborators. A new approach to problems involving random phenomena has been developed based on the concept of random distribution. This makes it possible to attack problems in nonlinear fluid dynamics, in particular the Euler and Navier-Stokes system, where only weak topologies can be used due to the lack of suitable a priori bounds. An original theory of compressible fluid flows driven by stochastic forcing has been built up practically from scratch and published in a compact form of a research monograph by E. Feireisl, D. Breit, and M. Hofmanová. It is the first monograph dealing with compressible viscous fluids driven by stochastic forcing. It includes a complete existence theory for problems involving the compressible Navier-Stokes system driven by multiplicative noise in time. Besides, the issues of weak-strong uniqueness, singular limits, and existence of stationary solutions are discussed. A completely new approach to stochastic PDEs is developed based on the concept of random distributions and a nontrivial generalization of the Skorokhod representation theorem is developed. Other results in this direction include the existence theory in the framework of weak solutions, the weak-strong uniqueness principle in the weak (in law) stochastic form, theory of singular limits, among others [Commun. Math. Phys., 2017, 0471015; Arch. Ration. Mech. Anal. 2016, 0462478; De Gruyter, 2018, 0485802; Probab. Theory Relat. Fields, 2019, 0506253].

Another subject of interest is related to the motion of a chemically reacting mixture or of a two-component mixture through a rigid porous medium. The research was done by E. Feireisl, H. Petzeltová jointly with P. Takáč (Universität Rostock), J. Mikyška (Czech Technical University) and D. Hilhorst (Université Paris-Sud), [J. Evol. Equ., 2016, 0457704; Springer, 2017, 0482840].

Numerical analysis in the framework of compressible fluids was also one of the topics studied by E. Feireisl. His aim was to derive a new approach to numerical problems in fluid mechanics as a nonlinear analogue of the celebrated Lax equivalence principle: stability + consistency = convergence. The method is based on: identifying a limit of a numerical scheme as a very general object – dissipative solution of the problem; applying the weak-strong uniqueness principle in the class of dissipative solutions; deducing strong convergence by careful analysis of the dissipation and concentration defect. Fundamental results have been obtained in collaboration with M. Lukáčová (Mainz) and H. Mizerová [Numer. Math., 2020, 0521527; Found. Comput. Math., 2018, 0489965]. Numerical methods in the dynamics of compressible viscous fluids were also studied by R. Hošek who is another former PhD students of E. Feireisl. His main contribution was related to discrete properties of meshes and analysis of convergence [SIAM J. Numer. Anal., 2016, 0463984; IMA J. Numer. Anal., 2019, 0509632]. Numerical analysis in the case of low Mach number limit was investigated by E. Feireisl, Š. Nečasová, M. Lukáčová (Universität Mainz) A. Novotný (Université de Toulon) and B. She [Multiscale Model. Simul., 2018, 0485868].

*Incompressible fluids*

Important questions related to finer qualitative properties of incompressible Navier-Stokes equations under various boundary conditions have been studied by J. Neustupa and his collaborators. Studying the structure of the set of steady solutions, they proved the existence of a strong-weak solution of the steady Bénard problem in a two-dimensional quadrangular cavity, heated/cooled on two opposite sides and thermally insulated on the other sides, and characterized the structure of the solution set [Nonlinear Anal., Theory Methods Appl., 2015, 0444159]. Similar results were obtained for a generalized Newtonian fluid in a bounded 2D or 3D domain [Nonlinear Anal., Real World Appl., 2019, 0492748]. Stability or instability of the flow is another topic of interest. In [J. Math. Fluid Mech., 2016, 0457135], it was shown that the question of stability of a steady incompressible Navier-Stokes flow in a 3D exterior domain is essentially a finite-dimensional problem and a sufficient condition for stability was formulated in terms of position of eigenvalues, regardless of the presence of an essential spectrum touching the imaginary axis. Similar questions in connection with a steady flow around a rotating body have been solved (with G. P. Galdi) in [Springer, 2016, 0466778]. The robustness of a solution to the Navier-Stokes equations with Navier's slip boundary conditions with respect to perturbations has been studied by J. Neustupa and P. Kučera, see [Nonlinearity, 2017, 0473686] and [J. Math. Anal. Appl., 2018, 0489052]. J. Neustupa together with our postdoc H. Al Baba studied the interior regularity of the pressure in regions, where the velocity satisfies Serrin’s integrability conditions under the Navier-type boundary conditions. They have also shown that the obtained results play an important role in the procedure of localization.

In [Adv. Math. Phys., 2018, 0487356], J. Neustupa (with P. Penel) showed that regularity up to the boundary of a weak solution of the Navier-Stokes equation with generalized Navier's slip boundary condition can be deduced from estimates of integrability rate of at least one of the eigenvalues of the associated strain rate tensor.

The paper [J. Math. Fluid Mech., 2018, 0492100] shows how the question of local regularity of weak solutions around a space-time point is related to a Serrin-type condition, imposed on the positive part of a certain linear combination of the pressure and the second powers of the velocity components. The solvability of problems with artificial outflow boundary conditions, described by variational inequalities, is shown by J. Neustupa with S. Kračmar in [Math. Nachr., 2018, 0492103]. Incompressible fluid flow in domains with moving boundaries, particularly around moving bodies, are studied in [Birkhäuser, 2016, 0458908], where J. Neustupa (with P. Penel) proved the existence of a weak solution to the Navier-Stokes equations with Navier's boundary condition in a moving domain under natural conditions including collisions of the solid bodies. A survey of results regarding mathematical models of motion of a body in an incompressible viscous fluid is summarized in a chapter in Handbook of Mathematical Analysis in Mechanics of Viscous Fluids J. Neustupa with G. P. Galdi [Springer, 2018, 0502444]. The paper in [Analysis, 2015, 0446811] represents a contribution in the 2D case to the famous problem of existence of a steady solution to the Navier-Stokes equations in a multiply connected domain in a situation that the prescribed fluxes through components of the boundary can be arbitrarily large. It is shown that the problem is solvable only under additional assumptions on the data.

A problem of non-periodic case of homogenization for incompressible fluids was solved by E. Feireisl, Š. Nečasová and Y. Namlyeyeva in [Manuscr. Math., 2016, 0455159].

Stationary problems were studied by V. Mácha. In [Czech. Math. J., 2016, 0460394] he proved higher integrability of the solution to a Stokes-type problem with elliptic term depending on the pressure. The non-diagonal case was studied in [Nonlinear Anal., Theory Methods Appl., 2018, 0488525]. Analytical properties of the Oseen and Stokes operators under different boundary conditions were found by H. Al Baba [Math. Nachr., 2019, 0505197; Differ. Equ. Appl., 2019, 0505200] or by D. Medková [J. Math. Fluid Mech., 2018, 0497142; Z. Angew. Math. Phys., 2018, 0493780; Math. Nachr., 2018, 0487819; J. Differ. Equations, 2016, 0462882; Math. Methods Appl. Sci., 2016, 0458378]. The monograph [Springer, 2018, 0489141] by D. Medková summarizes fundamental properties of solutions of the Laplace equation and the Poisson equation under various boundary value problems: Dirichlet problem, Neumann problem, Robin problem, transmission problem, problems with cracks, obstacle problem, mixed problems and oblique derivative problem. A functional-analytical background for elliptic PDEs including the Hardy inequality and compact embeddings in weighted function spaces is the subject of interest of A. Kufner and his collaborators, see [Rend. Ist. Mat. Univ. Trieste, 2017, 0484228], [World Scientific 2017, 0485775], [Math. Nachr., 2017, 0469658].

*Fluid-structure interaction*

The topic was introduced to the team by Š. Nečasová. The results obtained jointly with a large number of collaborators include the motion of a self-propelled body in compressible fluids which was shown to admit a solution [Proc. R. Soc. Edinb., Sect. A, Math., 2016, 0458215; Math. Models Methods Appl. Sci., 2016, 0456889], the motion of a rigid body with a cavity filled with compressible fluid, the system stabilized with a permanent rotation [Arch. Ration. Mech. Anal., 2019, 0503655], the collision problem and the influence of boundary conditions [Nonlinear Anal., Real World Appl., 2017, 0463845]. The weak-strong uniqueness for this problem was proved in [J. Math. Phys., 2019, 0500493], and the existence of a strong solution was shown in [Topol.

Methods Nonlinear Anal., 2018, 0494445]. An analysis of the adiabatic piston problem was carried out in [Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 2018, 0490500]. Problems of fluid flow in domains with elastic boundary have been a subject of [Springer ,2016, 0458907; J. Math. Soc. Japan, 2016, 0456888]. The Navier-Stokes-Fourier system in a moving domain with prescribed velocity was proved to be solvable and the low Mach number limit was identified in [J. Math. Pures Appl., 2018, 0483129; Z. Angew. Math. Phys., 2018, 0493344; J. Evol. Equ. 2016, 0463210]. Also the motion of fluid around rotating and translating rigid bodies was studied in a series of papers [Discrete Contin. Dyn. Syst., 2017, 0468916; Z. Angew. Math. Phys., 2017, 0468917], [Math. Nachr., 2016, 0462088], [Discrete Contin. Dyn. Syst., 2016, 0447662], [Math. Bohem., 2015, 0445819] and in the monograph [Atlantis Press, 2016, 0465184].

Viscoelastic fluids have been investigated following the so-called generalized Peterlin model [SIAM J. Math. Anal., 2017, 0476960] and the diffusive Peterlin model [Nonlinear Anal., Theory Methods Appl., 2015, 0443537]. Decay properties of steady and time-periodic Navier Stokes were investigated by a former postdoc T. Nakatsuka, see [J. Elliptic Parabol. Equ., 2018, 0489054; Analysis, 2018, 0489400].

### *Mathematical modelling in solid dynamics*

Research in solid dynamics is represented in the team by J. Jarušek, P. Krejčí, and G. A. Monteiro. One research area within solid dynamics modelling is related to control problems and hysteresis and is coordinated by P. Krejčí. Hysteresis is for example a typical feature of multifunctional (piezoelectric or magnetostrictive) materials. Neglecting hysteresis may lead to a loss of accuracy in engineering applications. In addition, the principles of continuum thermodynamics impose severe restrictions on models describing the interaction of several fields of different physical nature (mechanical stress, temperature, electric and magnetic field). A simple magnetostrictive energy harvester was proposed in [Commun. Nonlinear Sci. Numer. Simul., 2016, 0459093] and the efficiency of the harvesting process was studied in dependence of the model parameters. Advantages and drawbacks of different hysteresis models in engineering applications were studied in [Physica B, 2016, 0457322]. Most models have the drawback that they exhibit weaker or stronger discrepancy with experiments at low fields. This is why a new hysteretic model for ferroelasticity was proposed in [ZAMM, Z. Angew. Math. Mech., 2016, 0460530] including a mean field feedback effect. It was shown that it not only improves the agreement with experiments, but it leads to a well-posed mathematical problem when combined with the full system of balance equations. In [Arch. Appl. Mech., 2019, 0504848], the model was applied to the problem of optimal harvesting strategy for a piezoelectric harvester. Inversion of hysteresis operators is a major issue in optimization problems. The existence of a Lipschitz continuous inverse to a temperature-dependent Preisach operator was proved together with G. Monteiro in [Discrete Contin. Dyn. Syst., Ser. B, 2019, 0504463] and this result was subsequently used in [Nonlinear Anal., Real World Appl., 2019, 0496181] for proving the well-posedness of the momentum and energy balance equations of a vibrating piezoelectric rod subject to temperature changes. Numerical tests for parameter identification in models for magnetostriction have been carried out in [Appl. Math. Comput., 2018, 0481156].

Fluid diffusion in partially saturated deformable porous media is another area where strong hysteresis effects are observed as a result of surface tension on the interface between liquid and gas in the pores. A new full thermodynamic model for the interaction between the penetrating fluid and a deformable porous medium was developed jointly with B. Detmann (Albers) from the Universität Duisburg-Essen. It includes a degenerate hysteretic pressure-saturation relation, where hysteresis is assumed to be of Preisach type in agreement with engineering observations. First results about the solvability of the system were published in [Math. Methods Appl. Sci., 2016, 0459879]. Heat exchange between the fluid and a viscoelastoplastic porous medium was included in [SIAM J. Math. Anal., 2016, 0467681], and the case that the fluid may undergo a phase transition (water freezing and melting with specific volume change, for example) was studied in [Math. Models Methods Appl. Sci., 2017, 0481815]. Long-time stabilization of solutions to degenerate diffusion problems with hysteresis was proved in the monograph *Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs* [Springer, 2017, 0482847].

Hysteresis memory appears naturally also in economic modelling. It was shown in [Discrete Contin. Dyn. Syst., Ser. B, 2015, 0447655] that discontinuous economic processes can be efficiently described by families of the so-called Prandtl-Ishlinskii hysteresis operators. A special problem of optimal trading strategies in financial markets was studied in [Math. Bohem., 2016, 0459263], and it was shown using the Kurzweil integral representation of the Prandtl-Ishlinskii operator that a financial crash can be related to the existence of a strong dominant trader in the game. The synergies between hysteresis and the Kurzweil integration theory make it possible to overcome the difficulties triggered by discontinuities in the time evolution. Analytic properties of Prandtl-Ishlinskii operators were studied in [Discrete Contin. Dyn. Syst. 2015, 0439234; Discrete Contin. Dyn. Syst., Ser. B, 2017, 0477756], and optimal control strategies were proposed in [NoDEA, Nonlinear Differ. Equ. Appl., 2015, 0446770; Int. J. Control, 2018, 0481832], and [SIAM J. Control Optim., 2016, 0461927].

Contact problems and fatigue constitute another topic of interest among the team members. J. Jarušek investigated the dynamic thermo-viscoelastic Signorini contact problem. The existence of a weak solution to such problem was proved via the penalization of the contact condition [Tatra Mt. Math. Publ., 2015, 0449361]. In [Nonlinear Anal., Real World Appl., 2016, 0459254] it is proved that the acceleration and contact force are time-dependent measures with an at most countable number of nonzero singular parts in the zones of contact, hence the velocity field over the plate suffers jumps with natural physical interpretations of their signs. The characterizations of solutions presented here for the first time can be extended to all solutions to all dynamical contact problems where the mass impenetrability is assumed. The paper [Math. Mech. Solids, 2018, 0490888] is devoted to the rational dynamic contact with limited (ahead prescribed) interpenetration in elastodynamics, including the limit case when the pressure becomes infinite on the contact surface. The existence of solutions is proved via maximal monotonicity argument. In all cases the solutions converge weakly to a solution of an associated Signorini problem.

The so-called third body is a thin layer between two moving bodies in contact and is formed by nanoparticles detached from the moving bodies. It determines substantially the dynamics of the motion and its properties have been a subject of a systematic engineering investigation. A mathematical model for the third body was proposed in [Math. Mech. Solids, 2018, 0488091] and was shown to lead to a well-posed problem. Heat exchange between solids in contact was studied in [Nonlinear Anal., Real World Appl., 2015, 0435823]. The main issue was to formulate properly the energy balance on the contact between the body and the obstacle in order to obtain a well-posed problem. An analysis of the so-called mass redistribution method for contact problems was carried out in [J. Comput. Appl. Math., 2019, 0491866]. A model problem for material fatigue with heat exchange and phase transition was derived and solved in [Discrete Contin. Dyn. Syst., 2015, 0439237].

### *Bifurcation problems*

Bifurcations in reaction-diffusion systems was the main subject of interest of M. Kučera and M. Väth with their external collaborators. A variational approach was developed for systems with special unilateral sources and sinks of various types, and it was used for a description of a displacement of bifurcations of spatial patterns [Appl. Math., Praha, 2016, 0458817], [Nonlinear Anal., Theory Methods Appl., 2018, 0482022], [Nonlinear Anal., Theory Methods Appl. 2019, 0504264]. This is a non-variational problem, which is transformed to a variational one but even then it has an unusual structure and standard approaches cannot be used. It shows that the domain of parameters where bifurcation occurs is always enlarged or reduced by adding a unilateral source or sink into the equation for inhibitor or activator, respectively.

In this framework, also some abstract results were obtained. A variational characterisation of the largest eigenvalues of nonlinear positively homogeneous operators was given and together with a reduction of the system to a single equation it was used for the description of a displacement of bifurcations of spatial patterns in systems with non-smooth unilateral terms [Nonlinear Anal., Theory Methods Appl., 2018, 0482022; Nonlinear Anal., Theory Methods Appl., 2019, 0504264]. A modification of the well-known Crandall-Rabinowitz Bifurcation Theorem was proved for non-differentiable perturbations of smooth mappings and used for obtaining more detailed description of bifurcations of spatial patterns in systems with small non-differentiable terms [Springer, 2017, 0486946], [J. Math. Anal. Appl., 2018, 0481828]. In contrast to the classical case, two bifurcation half-branches can bifurcate from two different bifurcation points.

In cooperation with the Numerical Analysis team, theoretical results were completed by numerical experiments and interpretations in biology. The shape of patterns under the influence of sources and sinks described by non-differentiable terms and their smooth approximations was shown [Phys. Rev. E, 2017, 0477232].

In [Nonlinear Anal., Theory Methods Appl., 2016, 0458505], M. Väth proved the nonexistence of solutions for a large class of reaction-diffusion systems subject to Turing's diffusion-driven instability with sufficiently small unilateral perturbations. This has consequences for the topological degree of an associated mapping which in turn implies the existence of bifurcations in a parameter domain where no solutions would exist without the unilateral perturbations.

In [Ann. Mat. Pura Appl., 2017, 0470372] a necessary and sufficient condition was given for the compactness of the derivative of a map at some point. The result applies also if the “derivative” fails to be linear but is, for instance only positively homogeneous of some order.

### *Cooperation with industry*

There exists a many-year-long research cooperation between the team and industrial partner Doosan-Bobcat in the area of modelling and control of processes in production and optimization of loaders and excavators as a part of the programme Strategy AV21 coordinated by the Czech Academy of Sciences. Team members have organised regular workshops Mathematics for Industry (September 2017, October 2018, November 2019) in cooperation with engineers from the Doosan-Bobcat company. Upon their recommendation, the team have invited also M. Kaltenbacher from Vienna to join these activities. Among others, he delivered a lecture at the summer school Waves in Flows. Team members also took active part in the last joint workshop organised at the Doosan-Bobcat laboratories in November 2019 with participation of their highest Korean and US representatives discussing further cooperation.

The team participates at the EU-MATHS-IN.CZ, which is the Czech Network for Mathematics in Industry, a part of the European Service Network of Mathematics for Industry and Innovation.

## Research activity and characterisation of the main scientific results

### *Mathematical Logic*

In mathematical logic the team focused on proof complexity, bounded arithmetic, nonclassical logics and set theory.

*Proof complexity*

Proof complexity can be considered a part of logic, or a part of computational complexity theory, which belongs to theoretical computer science. Since the group working in this field is often using logical methods, proof complexity is included in logic in this report.

Perhaps the most interesting result proved by the team in proof complexity was P. Hrubeš' and P. Pudlák's exponential lower bound on the cutting-plane proofs of random formulas [FOCS 2017, 0483703]. Random formulas are conjectured to be hard for every proof system and for some weak proof systems this has been shown. The first such result, a lower bound for the resolution system, was proved by Chvátal and Szemerédi in 1988. For a long time, proving such a lower bound for the cutting-plane proof system seemed elusive.

The group solved two connected problems about the strength of approximate counting in proof complexity. P. Pudlák and N. Thapen proved width and size lower bounds on random resolution [Random resolution refutations, CCC, 2017, 0477098 & Comput. Complexity, 2019, 0504571]. This is a system that extends resolution by, roughly speaking, allowing the use of axioms which are true with high probability. Their argument used a new kind of switching lemma, which was then used by Kolodziejczyk and N. Thapen [Approximate counting and NP search problems, submitted to the Journal of Mathematical Logic, 2018] to solve the main open question in this area: showing that E. Jeřábek's full theory of approximate counting, measured by its provable NP search problems, is weaker than full bounded arithmetic.

P. Pudlák, V. Rödl, N. Thapen and a former postdoc in the team M. Lauria studied the problem of how difficult it is to prove that a graph has certain Ramsey property, namely, it does not have cliques and independent sets larger than $O(\log|V|)$. They proved a superpolynomial lower bound on resolution proofs of the tautologies expressing this property for all graphs with this property [Combinatorica 2017, 0474390]. This result is an example of the interdisciplinary nature of the work of the team: it connects logic, computational complexity, and combinatorics.

N. Thapen, with S. Buss, comprehensively studied [SAT, 2019, 0507739] the DRAT propositional proof system, and a range of related systems, from the point of view of proof complexity. DRAT is a relatively recent extension of resolution which is widely used to record refutations in practical SAT solving. In particular they showed the first lower bounds for the system DRAT without new variables. Working with Galesi and Kolodziejczyk [FOCS, 2019, 0518434], N. Thapen also discovered a new connection between two proof systems, showing that the space required to refute something in polynomial calculus is at least the square root of the width required to refute it in resolution, partially answering an old question about the systems. The construction used gives an entirely new way of proving lower bounds on space.

N. Talebanfard together with N. Galesi and J. Torán studied the structure of resolution proofs. One of their results is that with respect to width, regular resolution is optimal over Tseitin formulas, which means that regular proofs have the smallest with among all resolution proofs of Tseitin formulas [ACM Trans. Comp. Theory 2020, 0523857]. This answers a question posed by A. Urquhart at the workshop "Theory and Practice of SAT Solving", 2015.

E. Jeřábek [Ann. Pure Appl. Logic, 2017, 0464411] studied the proof complexity of implicational formulas in intuitionistic and superintuitionistic calculi. Previously known lower bounds on the lengths of proofs in these logics were generally based on variants of the feasible disjunction property, and as such seemed to rely in an essential way on tautologies using the disjunction connective. E. Jeřábek proved that disjunction is in fact not needed: there is a translation of intuitionistic formulas to purely implicational formulas that preserves up to a polynomial blow-up the lengths of proofs in basic intuitionistic calculi. On the other hand, intuitionistic proofs of implicational tautologies can be simulated by purely implicational proofs. Similar results also hold to some extent for other superintuitionistic logics, and for fragments with more connectives.

One topic that P. Pudlák has studied for many years is connections between hardness of proving finite consistencies of first-order theories and computational complexity. This research goes back to a lower bound proved (but not published) by Harvey Friedman in 1979 and rediscovered by P. Pudlák a few years later, and a joint paper with J. Krajíček from 1989. The article *Incompleteness in the Finite Domain* [Bull. Symb. Logic, 2017, 0487354] surveys former results and presents several new ones. More results have been obtained by his student E. Khaniki before he started the PhD studies at Charles University last fall [arXiv:1904.01362].

*Bounded arithmetic*

N. Thapen and coauthors A. Beckmann, S. Buss, S. Friedman and M. Muller developed a model for doing complexity theory in a set-theoretic setting, over arbitrary sets (not only finite). In a series of papers [Ann. Pure Appl. Logic, 2016, 0456152; Sets and Computations, 2017, 0476044; Computability, 2019, 0499289] they developed an analogue of polynomial time, showed that it has many different natural definitions and developed a weak set theory that works as a "bounded arithmetic" for it.

E. Jeřábek [J. Math. Logic, 2020, 0520996] studied the relationships between several common sufficient conditions for essential undecidability of first-order theories. In particular, he proved that there are theories that can represent all partial recursive functions, but cannot interpret Robinson's theory R. While this is a question of proof theoretic nature, the paper draws on a fruitful connection to tame model theory, namely classification-theoretic properties of the model completion of the empty theory. The argument involves a characterization of ∃∀ theories interpretable in existential theories.

Induction schemata without parameters and related induction rules are known in the area of strong fragments of arithmetic to have many interesting properties, such as conservation results and connections to local reflection principles. E. Jeřábek [Arch. Math. Logic, 2020, 0523579] undertook a systematic investigation of parameter-free induction axioms in the context of Buss's theories of bounded arithmetic, including parameter-free $\Pi^b$ schemata that were previously neglected. He obtained various results on inclusions between the fragments, conservation results, and equivalence of the fragments to reflection principles for quantified propositional proof systems. He also discovered a new witnessing theorem for unbounded consequences of $T^i$ and $S^i$ of a certain form.

*Nonclassical logics*

E. Jeřábek continued his work on unifiability and admissibility of rules with parameters in transitive modal logics. He made a comprehensive investigation of the computational complexity of the unifiability and inadmissibility problems, and showed that for a representative class of logics, they can be classified as being complete for one of the five classes $\Sigma_2^{\mathrm{exp}}$, NEXP, coNEXP, PSPACE, or $\Pi_2^p$. The paper was published in [Ann. Pure Appl. Logic, 2020, 0524632].

In classical universal algebra, a clone is a class of operations $B^n \longrightarrow B$ on a fixed base set $B$ that can be computed by circuits over a given gate set. Clones can be studied by their relational invariants using the clone–coclone duality. In reversible computing, we are similarly interested in sets of functions $B^n \longrightarrow B^n$ computable by reversible circuits using various gate sets. E. Jeřábek [Algebra Univers., 2018, 0489142] discovered a generalization of the clone–coclone duality that encompasses classical clones, closed classes of reversible operations, and more generally, classes of partial multifunctions $B^n \longrightarrow B^m$. Instead of relations, it employs invariants valued in partially ordered monoids.

A. Tabatabai, a PhD student in the team until 2018, has written 7 papers about various topics in logic: provability logic and arithmetical interpretations of modal logics [Provability Interpretation of Propositional and Modal Logics arXiv:1704.07677, Provability Logics of Hierarchies arXiv:1704.07678, Russellian Propositional Logic and the BHK Interpretation arXiv:1704.07679], bounded arithmetic [Computational Flows in Arithmetic arXiv:1711.01735], intuitionistic logic [Geometric Modality and Weak Exponentials arXiv:1711.01736], and substructural logics [Universal Proof Theory: Semi-analytic Rules and Craig Interpolation arXiv:1808.06256, Universal Proof Theory: Semi-analytic Rules and Uniform Interpolation arXiv:1808.06258]. The latter two are joint works with another PhD student R. Jalali. Since these papers are mostly long and he submitted them at the end of his studies, none has appeared in print so far.

*Set theory*

The main focus of the set-theory oriented members of the research team was the set theory of the reals, mainly the structure of filters and ideals on natural numbers. The main tools were inevitably the method of forcing and descriptive set theory. The researchers achieved a number of results, authoring or co-authoring about 20 papers (published or submitted). The work of the team was centered around a regular weekly research seminar at the Institute.

Let us mention the most important results in more detail. Probably the most intensively studied objects were ultrafilters on natural numbers and their combinatorial properties. Among these, the P-points are arguably of the most importance in several other areas of mathematics. However, their existence cannot be proved from the standard axioms of the set theory ZFC alone. D. Chodounský with O. Guzmán [Isr. J. Math., 2019, 0507734] introduced a new forcing method for building models without P-points and proved e.g. the following new results: There are canonical models without P-points; there are models without P-points with the size of the continuum arbitrarily large; there is an axiom which implies the non-existence of P-points.

B. Kuzeljević with D. Raghavan [J. Math. Logic, 2018, 0489985] answered a longstanding question of A. Blass as well as a question of N. Dobrinen and S. Todorčević by demonstrating that there may consistently exist a chain of ultrafilters, all of them P-points, longer than the size of the continuum, which is strongly increasing with respect to both the Tukey order and the Rudin–Keisler order.

A classical result of S. Todorčević characterizes selective ultrafilters as precisely ultrafilters generic on the poset on infinite sets over the model of $L(R)$. D. Chodounský (*Ideals and their generic ultrafilters*; to appear in Notre Dame J. Form. Log., with J. Zapletal) gave a new proof of this theorem, which generalized the result and provided a characterization of all ultrafilters generic on posets of the form $P(\omega)/\mathrm{fin}$ over $L(R)$.

An important tool to study the structure of the set filters or ideals is the Katětov order. Many combinatorial statements can be rephrased into the language of a certain Borel ideal being Katětov-above another Borel ideal. J. Grebík with M. Hrušák [Fundam. Math., 2020, 0518279] proved that there is no tall Borel ideal minimal among all tall Borel ideals. The tool for resolving this open question turned out to be a descriptive set theoretic method.

J. Grebík [Proc. Am. Math. Soc., 2017, 0475626] constructed a graph which has the same extension property as the Rado graph with a trivial automorphism group, and a complete metric space with the same extension property as the Urysohn space, yet its group of isometries is trivial.

### *Theoretical computer science*

The research area of circuit lower bounds is interesting, in the first place, as it poses mathematical challenges that are still beyond the reach for the known techniques; thus via addressing the challenging problems in the realm of circuit complexity, an investigator endeavours to seek for new methods and ideas. In a joint work with Meir, Weinstein and Wigderson [SIAM J. Comput., 2017, 0473043], D. Gavinsky worked on the problem of giving super-logarithmic lower bounds on the circuit depth via proving the direct sum conjecture, as suggested by Karchmer, Raz and Wigderson in 1995. This work develops a new approach that is based on information complexity and uses it to show that the KRW conjecture holds for the composition of a function with a universal relation. The work is a result of tight collaboration, and it is hardly possible to trace the origin of ideas to particular authors.

The problem of understanding the advantage of quantum computing over the classical one is one of the ways that can be used to understand the essence of quantum mechanics; assuming that the physical reality adheres to its description given by the quantum theory, this line of research can be viewed as aimed at exploring the core principles of the Universe. As of now, the most promising approach to studying quantum computing is, arguably, via investigating quantum communication complexity. D. Gavinsky's research interests include non-experimental demonstration of quantum supremacy in communication complexity. During the reported period he demonstrated [STOC, 2016, 0463615] a partial function with an efficient quantum simultaneous-messages protocol, but still no efficient classical two-way protocol. Later he presented [IEEE Trans. Inf. Theory, 2019, 0509379] a partial function that could be computed by a quantum protocol sending "bare" simultaneous messages of poly-logarithmic size, and whose classical simultaneous-message complexity is lower-bounded by a polynomial: this was the strongest known super-classical behaviour of quantum simultaneous-message protocols with respect to partial functions.

Probabilistic theory plays an important role in many of our works. Sometimes it even becomes the object of an investigation. In a joint work with S. Lovett, M. Saks and S. Srinivasan [Random Struct. Algorithms, 2015, 0434454], D. Gavinsky proved a strong Chernoff-like large deviation bound on the sum of non-independent random variables that have the read-k dependence structure. In their joint work [Commentat. Math. Univ. Carol., 2016, 0463333], D. Gavinsky and P. Pudlák addressed the question "How low can the joint entropy of $n$ $d$-wise independent discrete random variables be?", they have improved the previously known bounds and analysed some new regimes. In the work [Theory Comput Syst., 2020, 0531290] by the same authors, the famous notion of semi-random sources, also known as Santha-Vazirani (SV) sources, has been compared to its widely studied strengthening (strong SV-sources): a family of deterministic condensers has been constructed for the latter, whilst they are known not to exist for the former.

In automata theory, by adapting some properties from deterministic automata to nondeterministic automata, J. Komenda and T. Masopust defined new classes of very weak nondeterministic finite automata recognizing families of subregular languages (piecewise testable languages and -trivial languages) that are of interest in algebraic theory of languages, first-order logic (quantifier alternation hierarchy), database theory (XML databases and graph knowledge bases), and system theory (as the "simplest" deadlock-free systems). They investigated the complexity of fundamental questions of these models, including the inclusion, equivalence, and universality questions [Inf. Comput., 2017, 0476953].

In the theory of machine learning, P. Hrubeš with co-authors S. Ben-David, M. Shay, A. Shpilka and A. Yehudayov, proved undecidability of a certain problem about learning by showing that the answer depends on the Continuum Hypothesis [Nature Machine Inteligence, 2019, 0500071].

N. Talebanfard and P. Pudlák together with D. Scheder studied the performance of the fastest known algorithm for satisfiability of $k$-CNFs, called PPSZ algorithm. This is important in connection with the Strong Exponential Time Hypothesis that, roughly speaking, states that for $k$-SAT, the complexity approaches to that exhaustive search with going to infinity. They improved the estimate on how much time the PPSZ algorithm needs in the worst case [Tighter hard instances for PPSZ, ICALP 2017, 0476174].

### *Finite Combinatorics*

J. Hladký, together with Garbe, Hancock, and Pelekis, who were postdocs in the group in the evaluated period worked in extremal graph theory, random graphs, and graph limits. These are areas of discrete mathematics that have developed substantially in the last decade. Let us summarize the most important projects in the evaluation period.

In a series of four papers [SIAM J. Discrete Math., 2017, 0474810, 0474809, 0474808], in collaboration with J. Komlós, D. Piguet, M. Simonovits, M. Stein, and E. Szemerédi, J. Hladký approximately solved the Loebl-Komlós-Sós conjecture, which is an open conjecture from extremal graph theory from the 1990s. The most important part of their solution, which totals 200+ pages, is to circumvent the notorious limitation of the celebrated Szemerédi regularity lemma, so that the modified variant applies to sparse graphs. Most of the work on the problem was done prior to the evaluated period.

J. Hladký was working on packing conjectures of Gyárfás (1976) and Ringel (1963) and a related Graceful tree labelling conjecture of Rosa (1967). While these conjectures remain open, he proved their asymptotic versions (and some generalizations thereof), [Adv. Math., 2019, 0507738; Random Struct. Algorithms, 2020, 0524449], building on his previous work [Isr. J. Math., 2016, 0454288] which introduced modern probabilistic techniques to the field.

Most of J. Hladký's work in the last 4 years has been in the theory of limits of dense graphs, which is a recent theory developed by L. Lovász and his collaborators. He has been working in this field with postdocs F. Garbe, R. Hancock and C. Pelekis. In the evaluated period, they published 11 papers (published or arXiv preprints) in the area. The most important project of these is his work on related the weak* topology and the so-called cut distance. Let us give details. In the heart of the theory of dense graph limits is a fact that the space of graphons equipped with the cut distance is compact. This statement was first proven by L. Lovász and B. Szegedy in 2006 using combinatorial arguments. In [J. Comb. Theory, Ser. B, 2019, 0504394], an alternative proof is given which gives further deep insights. A more abstract version of this approach is worked out in [J. Comb. Theory, Ser. B, to appear, arXiv:1809.03797] from which a comprehensive theory emerged. The most important applications of these insights, worked out in [arXiv:1909.10987], is characterization "a connected graph is weakly norming if and only if it is step Sidorenko". Here, the concept of norming graphs is an emerging field in extremal graph theory with links to functional analysis, and is related to so-called Gowers uniformity norms in additive combinatorics.

C. Pelekis was mostly working on real-analytic counterparts from extremal combinatorics. He published 5 papers on the topic. For example, Sperner's theorem, which is one of the cornerstones of extremal combinatorics, gives a bound on the size of any antichain of subsets of a given set with respect to the inclusion relation. In [Isr. J. Math., 2020, 0532203], Pelekis obtains a counterpart of Sperner's theorem in the setting of antichains of $[0,1]^{\mathrm{n}}$. The "size" is measured using the Hausdorff measures.

F. Garbe also continued working on hypergraph packing problems with colleagues from the University of Birmingham using hypergraph regularity and the absorbing method. This led to an article about the structure of Hamiltonian graphs below the Dirac threshold stating necessary and sufficient conditions for 2-Hamiltonicity in dense 4-graphs, see Hamilton cycles in hypergraphs below the Dirac threshold [J. Comb. Theory, Ser. B, 2018, 0493792] and further still ongoing projects. Furthermore, Garbe investigated minimal percolating sets in certain bootstrap percolation processes in Contagious sets in a degree-proportional bootstrap percolation process [Random Struct. Algorithms, 2018, 0496182].

### *Control theory*

In control theory of logical discrete-event systems with a modular structure and a global specification, the aim is to locally compute a set of controllers. The emphasis of the research of J. Komenda and T. Masopust has been put on computationally efficient approaches for verification and control of systems with a large number of components, such as multi-level hierarchical control, and on sufficient conditions that guarantee equivalence between a computationally efficient modular control synthesis and a computationally intractable monolithic control synthesis. In their recent work, the previously known sufficient conditions for maximal permissiveness of modular control synthesis have been weakened and generalized in several directions [IEEE CDC, 2016, 0470380].

In the context of verification, we investigated the question of how to decompose a global specification to the corresponding local components. This problem is in general difficult. Even the problem to verify whether a given decomposition is correct is a PSPACE-complete problem. However, in a previous work, J. Komenda and T. Masopust have shown that under some reasonable and constructive assumptions the verification can be done in polynomial time [Discrete Event Dyn. Syst., 2015, 0443131]. They have continued this work by further investigating the decomposability verification problem and suggested a new approach where instead of checking decomposability for modules, which is PSPACE-complete, decomposability with respect to a carefully designed set of pairs of modules is checked, which may reduce the complexity. The main interest has been in an automatic design of such sets. The problem is partially solved in [IEEE Trans. Autom. Control, 2019, 0502163].

J. Komenda and T. Masopust further investigated the complexity of verification of some state-estimation properties, including detectability, opacity, and diagnosability, for modular and infinite-state systems modeled by automata and Petri nets, and answered several open problems [Automatica, 2019, 0504048].

In the context of control, they investigated the algorithmic complexity of existing techniques constructing supervisors. Known algorithms computing the supervisors based on the construction of infimal observable superlanguages are exponential and the existence of a polynomial algorithm was unknown. J. Komenda and T. Masopust solved the problem by showing that there is no algorithm with polynomial-time complexity computing and constructing the supervisor and developed a new approach to improve the algorithmic complexity. They have shown that our algorithm is tight from the point of view of algorithmic complexity [IEEE Trans. Autom. Control, 2018, 0484221].

In the theory of timed discrete-event systems, J. Komenda and T. Masopust have investigated modeling, analysis, and control of timed extensions of Petri nets by algebraic methods relying on dioid algebras such as max-plus and min-plus algebras. Among the main achievements in this area they mention compositional modeling of safe timed Petri nets by (nondeterministic) max-plus automata [Discrete Event Dyn. Syst., 2015, 0443129], modeling of bounded timed Petri nets with race semantics by deterministic max-plus automata [Discrete Event Dyn. Syst., 2016, 0460479], consistency analysis of P-time event graphs using linear inequalities in max-plus and min-plus algebras [IFAC World Congress, 2017, 0480320], synthesis of optimal supervisors for max-plus automata [Int. J. Control, 2015, 0450641], and improvements of sufficient conditions for determinization of max-plus automata [Discrete Event Dyn. Syst., 2020, 0522961].

## Research activity and characterisation of the main scientific results

In the period 2015–2019 the team members published 4 books, 40 journal papers, 19 papers in conference proceedings, 9 papers in domestic popularization journals, and they were editors of 10 conference proceedings. Below we present their main research achievements.

### *Finite element method and adaptivity*

*The theory of the finite element method* has traditionally represented an integral part of the activities of the team. The research during the evaluated period focused mainly on various geometrical aspects related to the generation and adaptive refinement of finite element meshes as well as to $hp$-adaptivity and parallelization of the mesh adaptive algorithm.

M. Křížek, with his coauthors, published several interesting results about geometric aspects of $n$-dimensional simplices. Together with J. Brandts and A. Cihangir from the University of Amsterdam, he investigated the behaviour of the sum of dihedral angles of nonobtuse $n$-dimensional simplices. They generalized the widely known fact that the sum of all three angles of a triangle is equal to two right angles and derived tight bounds on the dihedral angle sums for the subclass of nonobtuse simplices. All the dihedral angles of such simplices are less than or equal to the right angle. The main conclusion is that when the spatial dimension $n$ is even, the range of dihedral angle sum of nonobtuse simplices is $n$ times smaller than the corresponding range for arbitrary simplices. When $n$ is odd, it is $(n-1)$ times smaller. Several important applications of these results were published in M. Křížek et al. [Appl. Math. Comput., 2015, 0447563].

M. Křížek, together with J. Brandts in [J. Geom., 2019, 0508906], defined the so-called dual simplex of an $n$-simplex and proved that the dual of each simplex contains its circumcenter, which means that it is well-centred. They examined the similarity of a simplex and its first or second dual as well as the iterative application of taking the dual. For triangles, this iteration converges to an equilateral triangle for any initial triangle. For tetrahedra, they investigate the limit points of period two, which are known as isosceles or equifacetal tetrahedra.

Furthermore, in the proceedings paper [ENUMATH, 2019, 0499410], J. Brandts and M. Křížek study simplicial vertex-normal duality with applications to well-centred simplices. In particular, they investigate the relation between the set of $n+1$ vertices on an $n$-simplex $S$ having $S^{n-1}$ as its circumsphere and the set of $n+1$ unit outward normals to the facets of $S$. These normals can in turn be interpreted as vertices of another simplex $\hat{S}$ that has $S^{n-1}$ as its circumsphere. The authors consider the iterative application of the map that takes $S$ to $\hat{S}$, study its convergence properties, and in particular examine its fixed points. They also prove some statements about well-centred simplices in the above context.

Analyzing the general convergence properties of the finite element method, M. Křížek together with A. Hannukainen from the Aalto University of Helsinki and S. Korotov from the Western Norway University of Applied Sciences generalized the maximum angle condition of J. L. Synge for triangular finite elements to higher-dimensional finite elements [Appl. Math., Praha, 2017, 0471867]. In particular, they proved optimal interpolation properties in Sobolev norms of linear simplicial elements that degenerate in a certain way.

Spurious oscillations of numerical approximations caused by discontinuities and steep gradients of the (unknown) solution represent a widely studied and very challenging problem. P. Kůs, together with his long-term collaborators from the group of S. Badia from the International Centre for Numerical Methods in Engineering, Barcelona, Spain, developed an $hp$-adaptive algorithm for discontinuous Galerkin methods dealing with this problem. Taking advantage of the evolution of the gradient through the adaptive process, they developed a novel detector of discontinuities and steep gradients that marks the trouble-making elements for refinement and imposes their linear order. Applying the artificial viscosity to marked elements only controls the spurious oscillations and keeps the accuracy of the scheme. This result was published in [Comput. Methods Appl. Mech. Eng., 2016, 0465181].

Another challenging and currently heavily investigated problem is the efficient parallelization of mesh adaptive algorithms. In [Adv. Eng. Softw., 2017, 0475170], P. Kůs and J. Šístek presented their first result on this topic. In the context of the non-overlapping domain decomposition for the finite element method, they partition a space-filling curve to define well-balanced subdomains and analyse the influence of the arising non-standard meshes on the overall performance. The presented scalability for up to 2000 cores and more than a billion unknowns promises successful future applications.

The experience of the team with higher-order finite element methods was utilized by P. Kůs in [Appl. Math. Proc., 2015, 0450752], where he investigated convergence and stability of numerical solutions of highly nonlinear reaction-diffusion equations exhibiting the so-called Turing instability. He carried out numerical simulations of Turing pattern formation and confirmed the convergence of the method. However, for special choices of parameters close to bifurcation points, he found a very uncommon convergence behaviour.

### *A posteriori error estimates and numerical stability analysis*

A posteriori error estimates also belong to the traditional topics of the group, and even in the evaluated period, the team contributed with several new results. On the other hand, numerical stability was underrepresented in recent years, but in B. She the team gained an expert on this topic, especially in the context of the computational fluid dynamics.

In the field of a posteriori error estimates the team focused on guaranteed a posteriori error bounds for finite element solutions of both boundary value and eigenvalue problems. Among the main results in this direction belongs the paper [Comput. Methods Appl. Mech. Eng., 2019, 0505170] of T. Vejchodský and M. Ainsworth from the Brown University. The authors proposed a simple guaranteed a posteriori error bound for singularly perturbed reaction-diffusion problem and proved its robust local efficiency even in the singularly perturbed case, where the reaction term dominates the diffusion, the solution exhibits steep boundary layers, and finite element approximations oscillate.

In 2016 T. Vejchodský succeeded in the prestigious competition for Neuron Impuls research projects funded by the private Neuron Endowment Fund. His three-years-long project focused on the timely topic of guaranteed error bounds for eigenvalues and eigenfunctions of linear elliptic differential operators. In [Numer. Methods Partial Differ. Equations, 2018, 0489414], he studied bounds based on the flux reconstruction and compared their numerical efficiency with the classical Lehmann-Goerisch method and a new method based on an interpolation constant for Crouzeix-Raviart nonconforming finite elements. In his next work, he found that modern flux reconstructions developed recently for boundary value problems can be directly used to obtain sharp error bounds for eigenvalues by employing the traditional Lehmann-Goerisch method [J. Comput. Appl. Math., 2018, 0489966]. Further, with F. Bozorgnia and M. Seyyedi, he investigated computational methods for nonlinear eigenvalue systems involving $p$-Laplacian [Appl. Numer. Math., 2019, 0505711].

The challenging problem of a posteriori error bounds for computed eigenfunctions was addressed by T. Vejchodský and X. Liu from the Niigata University, Japan, at the end of the evaluated period and published in the preprint [arXiv:1904.07903]. They solved the ill-posedness of eigenfunctions for multiple and tightly clustered eigenvalues by considering the corresponding spaces of eigenfunctions and proposed an upper bound on the distance between this space and its approximation. The derived bound may be useful in practice because it is easy to compute just from the approximate eigenfunctions and two-sided bounds of eigenvalues.

The stability and convergence analysis of numerical methods became a new direction in the research focus of the team since B. She has joined the team. He together with R. Hošek from the University of West Bohemia in Pilsen, investigated a finite difference scheme for compressible Navier-Stokes equations in higher dimensions [J. Numer. Math., 2018, 0495265]. They designed a scheme on a staggered grid utilizing the upwind technique to achieve stability. They analyzed this natural scheme in detail and proved, for instance, the strict positivity of density, mass conservation, stability of energy, and the overall consistency of the scheme for a whole range of physically relevant adiabatic exponents. In addition, the authors prove the existence of numerical solutions of coupled difference equations.

The following results stem from the interest of B. She in the complex behaviour of viscoelastic polymeric fluids. In collaboration with M. Lukáčová-Medviďová from the University of Mainz, he successfully attacked the problem of instability of numerical schemes for high Weissenberg number. In the valuable contribution [Internat. J. Numer. Methods Fluids, 2016, 0461727] they propose two characteristic line methods for the approximation of Oldroyd-B viscoelastic fluid that are stable even for high Weissenberg number. They proved the energy stability of the schemes theoretically and validated it by numerical experiments. Interestingly, this result can also be applied to the simulation of viscoelastic fluids such as chemical-polymers.

B. She with H. Mizerová from the Comenius University in Bratislava analyzed a conservative scheme for the Fokker-Planck equation with applications to viscoelastic polymeric fluids [J. Comput. Phys., 2018, 0492752]. In particular, they concentrated on the motion of infinitely extensible polymeric fluids, which involves not only bounded physical domain but also the infinite phase space. They propose a new numerical scheme based on space-splitting that employs the Lagrange-Galerkin and Hermite spectral method for a higher dimensional Fokker-Planck equation in a bounded and unbounded domain, respectively. To demonstrate its performance, they present several numerical experiments and show the convergence in mesh points. They also prove the conservation of discrete mass and illustrate it by numerical experiments.

Finally, it is worth to mention the survey paper [Neural Net. World, 2016, 0461444] of M. Křížek, J. Brandts from the University of Amsterdam, and Z. Zhang from the Wayne State University in Detroit about computational paradoxes for various unstable numerical schemes. Out of three major sources of errors, namely modelling error, discretization error, and rounding errors, they concentrate on the rounding errors and present 10 pathological examples of numerical calculations.

### *Numerical linear algebra and matrix theory*

As it was already noted, numerical linear algebra is one of the subfields that were not adequately represented in the main scientific directions of the team in the past. The scope thus has been extended with the expertise in such fields as rounding error analysis of various matrix decompositions and orthogonalization schemes, and analysis of numerical behaviour of iterative methods for solving large-scale systems of linear equations, in particular the understanding of rounding error effects such as convergence delay and maximum attainable accuracy of approximate solutions computed in finite precision arithmetic.

In the field of iterative methods, enormous attention was devoted in the previous two decades to the accuracy of widely used GMRES method, other residual minimizing Krylov subspace methods and short-term conjugate gradient-type methods. These papers mainly considered solving large-scale numerically nonsingular systems.

The paper by M. Rozložník et al. [SIAM J. Matrix Anal. Appl., 2018, 0490058] contains one of the first results achieved recently on the numerical behaviour of the GMRES method for solving singular linear systems. For two particular classes of singular matrices, it is shown that the accuracy of the GMRES may deteriorate due to the inconsistency of the system, the distance of the initial residual to the nullspace, and the principal angles between the ranges of the matrix and its transpose. This is a principal difference from the behaviour of the GMRES on non-singular systems.

Recent activity in the research field of iterative methods concerns the numerical stability analysis of pipelined Krylov subspace methods. The work of M. Rozložník et al. [SIAM J. Sci. Comput., 2018, 0495023] reflects a renewed interest in pipelined conjugate gradient methods that came with the advent of massively parallel computer architectures. This paper recalls early developments embedding them into historical context, identifies main factors determining numerical instabilities and presents a methodology for their full-scale analysis. The design of efficient implementation thus requires not only the optimization of run-time per iteration but also understanding of its behaviour in finite precision arithmetic. This paper results from the long-term cooperation of the well-established numerical linear algebra group located in Prague, including E. Carson, who was that time at the Courant Institute in New York.

Another important topic investigated by the team is the solution of saddle-point problems. Large-scale linear systems of saddle-point type arise in a wide variety of applications throughout computational science and engineering. Due to their indefiniteness and poor spectral properties, these systems represent a significant challenge. In the previous period, M. Rozložník was interested in the analysis of inexact saddle point solvers [SIAM J. Numer. Anal. 2015, 0444138]. His recently published book [Nečas Center Series. Cham, Birkhäuser 2018, 0497275] discusses particular properties of such linear systems and a large selection of algebraic methods for their solution with emphasis on iterative methods and preconditioning. It covers the state of the art theory on the solution of saddle problems that can be found scattered in the literature. It also contains the original research results achieved by the author and his colleagues, namely V. Simoncini, P. Jiránek, Z. Z. Bai, A. Smoktunowicz, M. Tůma, and J. Maryška. Theoretical results are complemented with the case study of a potential fluid flow problem in a real-world application.

The research interests of the team include also rounding error analysis of direct methods and orthogonalization schemes that certainly represent the building blocks in linear algebra and matrix computations. Numerical stability of various orthogonalization schemes has attracted a lot of attention, including the contributions on the classical and reorthogonalized Gram-Schmidt algorithm, which is an important approach in the context of high-performance computing. Based on the relation of the approximate inverse preconditioning to generalized orthogonalization schemes, several results on orthogonalization with respect to non-standard inner products were published in [SIAM J. Matrix Anal. Appl. 2015, 0399416] and [SIAM J. Sci. Comput. 2016, 0456004]. In addition, through the connection of the so-called J-orthogonal matrices to the class of G matrices, the team was interested in matrix theory which led to the collaboration of M. Rozložník with M. Fiedler and later with F. J. Hall et al., see [Spec. Matrices 2017, 0480731]. The orthogonalization schemes with respect to bilinear forms are related to structured eigenvalue problems such as symplectic eigenvalue problems in a number of important applications. The analysis of structure-preserving transformations based on the Cholesky-like decomposition of skew-symmetric matrices was given in [Linear Algebra Appl. 2016, 0456732]. A relevant reference for this part of the research can also be the dissertation of M. Rozložník: *Gram-Schmidt orthogonalization in presence of rounding errors,* 231 pp., submitted in 2019 for the scientific degree of Doctor Scientiarum (DSc.).

During 2016–2018, J. Šístek contributed to the open-source numerical linear algebra library PLASMA. He was responsible for the parallel implementation of the QR and communicationavoiding QR algorithms in the new version of the library based on tasks with dependencies of the OpenMP standard. See the project webpage https://bitbucket.org/icl/plasma.

More in the direction of the matrix theory, M. Křížek and J. Brandts in [Czechoslovak Math. J., 2016, 0463394] developed a finite and exact algorithm to factorize cp-rank-3 completely positive matrices. A symmetric positive semi-definite matrix $A$ is called completely positive if there exists a matrix $B$ with nonnegative entries such that $A = BB^{\mathrm{T}}$. If $B$ is such a matrix with minimal number $p$ of columns, then $p$ is called the cp-rank of $A$. Failure of this algorithm implies that $A$ does not have cp-rank 3. The authors showed, for example, that there are no three nonnegative polynomials of degree at most four that vanish at the boundary of a bounded interval and are orthonormal with respect to a Sobolev $H^1$-inner product.

***Domain decomposition methods and high performance computing***

The team has important contributions in the field of domain decomposition methods and high performance computing. These activities also include implementation of parallel iterative solution techniques applied to various problems, e.g. numerical solution of incompressible Navier-Stokes equations and subsurface flow modelling, hence combining the expertise from all topics discussed in the previous paragraphs.

J. Šístek together with J. Březina and B. Sousedík from the University of Maryland published a paper [Numer. Linear Algebra Appl., 2015, 0450643] on Balancing Domain Decomposition by Constraints (BDDC) for a mixed-hybrid formulation of flow in porous media with combined mesh dimensions. This remarkably comprehensive paper focuses on large-scale simulations of subsurface water flows described by Darcy's law. It proposes the BDDC domain decomposition for fractures and presents numerical analysis showing how the linear system and the preconditioner lead to positive definite matrices. Parallel numerical experiments show scalability on up to 1024 CPU cores for both benchmark and real-life geoengineering problems with fractures. In [Proc. PANM, 2015, 0473830], M. Hanek, J. Šístek, and P. Burda from the Czech Technical University explored the applicability of the BDDC method to nonsymmetric saddle-point algebraic problems arising from numerical simulations governed by the Navier-Stokes equations. One step of BDDC was applied as the preconditioner for the BiCGstab method. Together with E. Stach, the authors have applied the solver to simulations of oil flow in hydrostatic bearings in [Proc. Fluid Mech., 2018, 0499835].

J. Šístek together with M. Čertíková and P. Burda published the paper [Proc. PANM, 2015, 0443847], where several choices of interface weights for domain decomposition methods are discussed. Two new versions of scaling are proposed and compared on model problems of the Poisson equation and linear elasticity in 3D. Problems with jumps in coefficients of material properties both for regular and irregular interfaces between subdomains are considered.

J. G. Calvo [Math. Models Methods Appl. Sci., 2018, 0490551] introduced a new extension operator for virtual coarse space which can be used in domain decomposition methods for nodal elliptic problems in two dimensions. In particular, a two-level overlapping Schwarz algorithm is considered and a bound for the condition number of the preconditioned system is obtained. This bound is independent of discontinuities across the interface. The extension operator saves computational time compared to previous studies, where discrete harmonic extensions are required and it is suitable for general polygonal meshes and irregular subdomains. Numerical experiments with regular and irregular polygonal elements and with subdomains obtained by a mesh partitioner verify the developed theoretical results.

J. Šístek together with F. Cirak from the University of Cambridge in [Comput. Fluids, 2015, 0448127] discuss aspects of implementation and performance of parallel iterative solution techniques applied to low Reynolds number flows around fixed and moving rigid bodies. The incompressible Navier-Stokes equations are discretized with Taylor-Hood finite elements in combination with a semi-implicit pressure-correction method. The resulting sequence of convection-diffusion and Poisson equations are solved with preconditioned Krylov subspace methods. To achieve overall scalability they consider new auxiliary algorithms for mesh handling and assembly of the system matrices. They compute the flow around a translating plate and a rotating insect wing to assess the scalability properties of the developed solver. The largest considered unstructured meshes led to around 3 billion unknowns and were solved on 65 thousand cores. The authors find that almost perfect scaling can be achieved with a suitable Krylov subspace iterative method, like conjugate gradients or GMRES, and a block Jacobi preconditioner with incomplete LU factorization as a subdomain solver. In addition to parallel performance data, they provide new highly-resolved computations of flow around a rotating insect wing and examine its vortex structure and aerodynamic loading.

This research has also led to the paper [Proc. Fluid Mech., 2015, 0441967], where J. Šístek presents a simulation of a benchmark problem of incompressible viscous flow around a sphere at Reynolds number 300 and the results are compared with literature.
The advances in domain decomposition methods are implemented in the open-source domain decomposition library BDDCML. J. Šístek has continued in its development as the main developer. The software library has been developed by 8 contibutors, and it is used by about 5 external software packages for solving algebraic problems arising from the finite element method. For details, visit the BDDCML webpage at the link https://users.math.cas.cz/~sistek/software/bddcml.html.

### *Applications of numerical methods*

The team is naturally interested in applications of numerical methods. Its members made several important contributions in selected applications, such as vortex identification in fluid mechanics, stochastic modelling of biochemical processes in cells, and general relativity, astrophysics and cosmology.

Although identification and visualization of vortices in three dimensional flow fields has been studied for several decades, the increasing accessibility and resolution of such data from flow simulations still presents a challenge. Consequently, existing methods are revisited and new methods are proposed, such as the recently proposed vortex vector (Rortex).

J. Šístek together with V. Kolář [AIP Adv., 2019, 0510373] considered a model vortex with a uniaxial stretching coupled with an inevitable uniform radial contraction for incompressible flow to present a straightforward comparison of the stretching response for several popular vortex-identification criteria and Rortex. The stretching sensitivity of the examined schemes significantly differs and reopens the persisting vortex-identification problem that the requirement of orbital compactness of the motion inside a vortex contradicts with the allowance for an arbitrary axial strain.

The same authors in [Int. J. Aerospace Engrg., 2019, 0503124] presented a new analysis of the vortex-identification Q-criterion and its recent modifications. In this unified framework based on different approaches to averaging of the cross-sectional balance between vorticity and strain rate in 3D, new relations among the existing modifications were derived. In addition, a new method based on spherical averaging was proposed. It is applicable to compressible flows, and it inherits a duality property which allows its use for identifying high strain-rate zones together with vortices.

J. Šístek and V. Kolář extended the earlier concept of the average co-rotation of infinitesimal radial line segments near a point to the case of contra-rotation [J. Phys., 2017, 0474197]. They introduced the tensor of the contra-rotation and averaged it over “all planar cross sections” going through the examined point. Both the average contra-rotation and co-rotation, representing shear-free quantities, were applied to describe a complex flow structure.

The same authors proposed earlier [AIAA J., 2015, 0445926] another 3-D modification of the vortex-identification Q-criterion, labeled $Q_M$-criterion. It is based on the corotational and compressibility arguments, and in this regard it represents a one-way extension from 2-D to 3-D concept. The $Q_M$-criterion was successfully applied to different flow situations: it closely reproduces the results of some widely used methods in regions of low external shear and performs better in regions of strong shearing similarly as the average-corotation method.

The new advanced vortex identification methods are made available through the open-source Vortex Analysis library (VALIB) developed by J. Šístek. See the webpage https://users.math.cas.cz/~sistek/software/valib.

Another direction of applications is motivated by mathematical biology and (bio)chemistry. T. Vejchodský exploited his contacts with the mathematical biology group from the University of Oxford on solving higher-dimensional problems appearing in biochemical applications. The main topic was the application of tensor methods for the solution of high dimensional problems with the vision of applications in stochastic modelling of biochemical processes in living cells. Within this topic, the following two results were published: T. Vejchodský et al. [J. R. Soc. Interface, 2015, 0444817] and [Phys. Rev. E, 2015, 0443354].

Within the same group, T. Vejchodský together with the PhD student T. Plesa and his supervisor R. Erban constructed a two-dimensional chemically plausible system exhibiting exotic bifurcations in both the deterministic and stochastic case. These results were published as a book chapter [Multiscale Model. Numer. Method Comput. Cellular Biol., 2017, 0481366] and paper [J. Math. Chem., 2016, 0463647].

T. Vejchodský is the member of the Mathematical biology seminar organized by M. Kučera from the Evolutionary Differential Equations team of the Institute. The interdisciplinary cooperation of the biologist F. Jaroš, mathematical analyst M. Kučera, numerical analyst T. Vejchodský and his PhD student V. Rybář initiated at this seminar resulted in the paper [Phys. Rev. E, 2017, 0477232] on the phenomenon of Turing patterns formation.

Another group of applications aims at astrophysics and cosmology. The standard cosmological model is based on the normalized Friedmann ordinary differential equation which was derived under excessive extrapolations from Einstein's partial differential equations, which are not scale invariant and are experimentally verified on much smaller scales. In the paper [Gravit. Cosmol., 2016, 0461447], M. Křížek and L. Somer from the Catholic University of America explain why these extrapolations are incorrect, why the unrestricted use of the term "verified" is questionable, and why dark matter may exist only by definition.

According to the standard cosmological model, our universe needs a significant amount of dark matter – about six times that of the usual baryonic matter, besides an even larger amount of dark energy. But to date, both dark matter and dark energy have remained conceptually delusive, without concrete evidence based on direct physical measurements. In [Gravit. Cosmol. 2018, 0496178] M. Křížek presented ten counterarguments showing that such a claimed amount of dark matter can be a vast overestimation and does not conform to reality.

The maximum mass of a neutron star is about three solar masses. In this case the radius of such neutron star is approximately equal to the Schwarzschild radius. Adding a small amount of matter to this star, a black hole arises. Thus, its interior could contain a star with neutron or quark density just below the event horizon instead of the proposed point singularity. This idea was published by M. Křížek in [Astrophys. Space Sci., 2019, 0511328]. He also showed that the Hawking miniature black hole evaporation is improbable, since it would yield unrealistic mean mass densities.

Further, M. Křížek wrote two papers on cosmological problems with A. Mészáros from the Astronomical Institute of the Charles University [CSS2016, 0462143] and [CSS2018, 0493351]. Together with Y. V. Dumin from the Russian Academy of Sciences, he also edited two proceedings

[CSS2016, 0465745] and [CSS2018, 0500464] of conferences Cosmology on Small Scales held in 2016 and 2018 at the Institute.

M. Křížek was also interested in number theory. Together with L. Somer he studied Lehmer pseudoprimes and established some necessary and sufficient conditions for a given integer number to be a Lehmer pseudoprime, superpseudoprime, or Lehmer prime. The same authors also investigated properties of Lucas sequences whose discriminants have the same nonzero character modulo a prime number $p$ and whose periods modulo $p$ are the same. They also investigated the second order Lucas sequences with given initial conditions and characterized all non-defective solutions of a complete system of residue modulo $m$. Finally, they found all fixed points of particular Lucas sequences modulo $m$ and all cycles of length two. These results were published in [Fibonacci Quart., 2015a; 0441527, 2015b, 0447260; 2015c, 0450652; 2016, 0462797; 2017, 0477953] and [Integers, 2018, 0489233].