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Logic and Implication: An Introduction to the General Algebraic Study of Non-classical Logics

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1 Classical logic and correct reasoning

Since it was established in the Western tradition by Aristotle in the fourth century BC, logic has been concerned with correct reasoning, that is, with the study of the valid ways by which one can infer a proposition (the conclusion) from a set of previously given propositions (the premises). It is a formal science because it concentrates on inference patterns that are valid solely by virtue of their form, not of their content.

A very successful account of valid reasoning is the one known as *classical logic*, which has been proposed as the cornerstone of all science, rationality, rigorous knowledge, and reliable communication. It is a study of logical inference that stems from the seminal Aristotelian syllogistic presented in the *Prior Analytics* and from the analysis of propositions developed by the Stoics since the third century BC. Later, it was widely used and developed by medieval logicians, and finally obtained its contemporary presentation in the nineteenth century by the founders of modern mathematical logic (Augustus De Morgan, George Boole, Gottlob Frege, and others). Thanks to their works, today we know classical logic as a well-developed mathematical machinery that allows one to determine the validity of, allegedly, any given argument that can be analyzed in terms of the usual propositional connectives and quantifiers.

Propositional classical logic is usually syntactically given in a language with the following logical connectives: negation $\neg$, implication $\to$, disjunction $\vee$, conjunction $\wedge$, and equivalence $\leftrightarrow$. Semantically, it operates on the basic assumption of *bivalence*, i.e. each well-formed meaningful proposition must always be either true or false, and any other possibility is excluded. This is implemented by giving a mathematical semantics to all sentences by means of the two-valued Boolean algebra $\boldsymbol{2}$: an algebraic structure with two values—1 which stands for *true* and 0 for *false*—and operations for the connectives ($\neg^{\boldsymbol{2}}$ for negation, $\to^{\boldsymbol{2}}$ for implication, $\vee^{\boldsymbol{2}}$ for disjunction, $\wedge^{\boldsymbol{2}}$ for conjunction, and $\leftrightarrow^{\boldsymbol{2}}$ for equivalence) defined as:

a	$\neg^{\boldsymbol{2}} a$
1	0
0	1

a	b	$a \to^{\boldsymbol{2}} b$	$a \vee^{\boldsymbol{2}} b$	$a \wedge^{\boldsymbol{2}} b$	$a \leftrightarrow^{\boldsymbol{2}} b$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	1	1	0	0
0	0	1	0	0	1

The semantics of all formulas is established in terms of *evaluations* into $\boldsymbol{2}$, i.e. mappings from the set of all formulas to $\{0, 1\}$ that use the algebraic operations above to interpret connectives (e.g. $e(\varphi \to \psi) = e(\varphi) \to^{\boldsymbol{2}} e(\psi)$, where e is an evaluation). This gives a very specific character to the classical semantics with several features worth commenting on:

Truth-functionality: The value of any complex sentence is computed using only the *values* of its constituent parts.

Material implication: In particular, the value of an implicational formula $\varphi \to \psi$ depends only on the value of φ (its *antecedent*) and the value of ψ (its *consequent*), disregarding any mutual relations their meanings might have.

Preservation of truth: Material implication captures a certain intuition of *transmission* or *preservation* of truth. Indeed, if an implication is true, the truth of its antecedent must imply the truth of its consequent; in other words, a true implication can never have a true antecedent and a false consequent.

Order of truth: As a simple reformulation of the previous observation, if we think that the two truth-values are ordered ($0 < 1$), we have that in a true implication the value of the antecedent is less than or equal to the value of the consequent.

Notable tautologies: The following formulas are tautologies (i.e. true under any evaluation):

$$\begin{aligned} \text{Double negation elimination} \quad & \neg\neg\varphi \to \varphi \\ \text{Law of non-contradiction} \quad & \neg(\varphi \wedge \neg\varphi) \\ \text{Law of excluded middle} \quad & \varphi \vee \neg\varphi \\ \text{Contraction} \quad & \varphi \to \varphi \wedge \varphi \\ \text{Weakening} \quad & \varphi \to (\psi \to \varphi). \end{aligned}$$

Using evaluations one can define the fundamental logical notion of *semantical consequence*. That is, if Γ is a set of formulas and φ is a formula, we say that φ (the *conclusion*) is a semantical consequence of Γ (the *premises*) if any possible evaluation of formulas that makes all the formulas in Γ true (i.e. equal to 1), must also make φ true; equivalently: there is no evaluation under which all formulas in Γ are true and φ is false. In symbols, we write: $\Gamma \models_2 \varphi$. Again, as in the case of implication, this definition follows the idea of *preservation of truth*: if $\Gamma \models_2 \varphi$, the truth of Γ must be preserved in the truth of φ. In this way, classical logic manages to capture one possible precise notion of *correct reasoning*, that is: an argument that uses Γ as the set of premises and obtains φ as the conclusion is correct precisely when $\Gamma \models_2 \varphi$.

Moreover, besides mathematical semantics, propositional classical logic has also been given a wealth of proof systems; that is, formal calculi allowing us to derive conclusions from sets of premises, based purely on symbolic manipulation of formal sentences, regardless of any semantical interpretation.

Among them, *Hilbert-style axiomatic systems* are very elementary and have a strong theoretical interest, while they may be difficult to use when looking for particular derivations of some given formulas. Other kinds of systems are more versatile for practical purposes (e.g. natural deduction, sequent and hypersequent calculi, tableaux, resolution, etc.) and have given rise to algorithms for automated theorem proving, a necessary prerequisite for applications of logic in problems of computer science and artificial intelligence.

Hilbert-style calculi consist of a set of *axioms* (formulas that are stipulated to be always true and can be used at any moment) and a set of *inference rules* (that allow one to derive new formulas from those that one already has). For example, a typical rule to capture the formal behavior of implication is the well-known *modus ponens*:

$$\varphi, \varphi \to \psi \blacktriangleright \psi.$$

According to this rule, whenever we prove a formula φ and a formula $\varphi \to \psi$, we can automatically derive the consequent ψ, just by virtue of the kind of symbols (i.e. implication) that are being manipulated, regardless of any semantical interpretation.

By introducing a sufficient number of such formal rules, we have the necessary means to obtain a notion of *formal proof* in the Hilbert-style system, construed as a finite sequence of formulas. The starting point of any proof, besides the axioms, is a set of premises that are taken as hypotheses in a certain context. Additional elements are then conclusions of inference rules whose premises already appear in the proof. Given a set Γ of formulas and a formula φ, we write $\Gamma \vdash_{\mathrm{CL}} \varphi$ (CL standing for *classical logic*) to signify that, when taking Γ as premises, there exists a formal proof in which φ is the last obtained formula, i.e. the conclusion.

Other kinds of proof systems give rise to their corresponding notions of formal proofs. To stress the opposition with the semantical rendering of correct reasoning, we may say that proof systems are purely *syntactical* devices, i.e. inference machines that pay attention only to the syntactical form of the expressions they manipulate.

If proof systems are to make sense and be maximally useful from the theoretical and the applied viewpoints, they should be:

Sound: If $\Gamma \vdash_{\mathrm{CL}} \varphi$, then $\Gamma \vDash_{\mathbf{2}} \varphi$. That is, the proof system is not too strong to formally prove claims which are not correct.

Complete: If $\Gamma \vDash_{\mathbf{2}} \varphi$, then $\Gamma \vdash_{\mathrm{CL}} \varphi$. That is, the proof system is strong enough to obtain formal proofs of all correct claims.

Thus a sound and complete (syntactical) proof system is equivalent to—it captures exactly—the (semantical) classical notion of correct reasoning.

We are now about to introduce a wealth of non-classical logics described semantically (via classes of models) and syntactically (via proof systems). One of the main topics of this disertation is the study of under which conditions a given syntactical and semantical presentation coincide, i.e. describe the same logic. Such results are historically called *completeness theorems,* despite of the fact that they are formulated as equivalence ('if and only if') statements, not as a single implication as the name and the classical terminology would suggest. Therefore, from now on, we use in this context the term 'complete' in the sense of 'sound and complete'. As we almost exclusively work in the context when soundness is already established, there is no danger of confusion.

2 Beyond classical logic

There have been a number of claims that classical logic, for all its merits, is not capable of providing a satisfactory explanation of correct reasoning in all possible forms and contexts. Many have pointed at important shortcomings of an analysis that requires the strong assumptions we have listed before, including the oftentimes too limiting restriction to bivalent semantics, or the validity of the law of excluded middle which forces any proposition to be such that either itself or its negation is true (in a given interpretation). Such restrictions confine the classical logician to a narrow set of connectives with several obvious limitations, including among others the following:

- The built-in truth-functionality imposes a strong restriction on expressive power. Indeed, it excludes intensional (i.e. non-truth-functional) contexts such as those given by modalities: necessity, possibility, propositional attitudes, etc.
- The bivalent connectives can have a rather unnatural behavior, as in the case of material implication, which takes as indisputably true any implication that has a tautology in the consequent or a contradiction in the antecedent.
- Bivalence, the law of excluded middle, and contraction cause very serious problems when confronted with some critical logical puzzles, such as self-reference paradoxes or the *sorites* paradox in the analysis of vagueness.
- The law of excluded middle and double negation elimination allow for *non-constructive* proofs by contradiction in mathematical reasoning, which has been seen as a commitment to a Platonist view of mathematics (a view that takes mathematical objects as abstract entities with independent existence).

The discussion about the limits of classical logic has given rise during the last century to a plethora of alternative non-classical logical systems, based on a wide range of different motivations (to which we have only slightly hinted). Some of them have been proposed to amend alleged deficiencies of classical logic in certain reasoning contexts (e.g. intuitionistic logic for constructive mathematical reasoning, or relevant logics when material implication is not considered adequate). Other non-classical logics have been developed as useful *technical devices* for a finer analysis of reasoning (e.g. fuzzy logics for reasoning with graded predicates or paraconsistent logics for reasoning in the presence of contradictions) or to model other phenomena (e.g. linear logics for computational tasks or Lambek logic for linguistic analysis). Finally, many others have been defined and studied out of sheer mathematical curiosity.

Formally speaking, some of these alternative non-classical propositional logics are expansions of classical logic with new syntactical devices (such as modalities) that ensure a higher expressive power, while others invalidate some problematic classical truths. Let us briefly introduce three classes of such logical systems (which will play an important role in this dissertation), stressing the main aspects in which they deviate from the classical paradigm and exemplifying the difference in the semantics of implication.

2.1 Many-valued logics

The twentieth century witnessed a proliferation of logical systems which, though still truth-functional, deviate from classical logic by having an intended algebraic semantics with more than two truth-values (for a historical account see e.g. [19]). Prominent examples are 3-valued systems like Kleene's logic of indeterminacy and Priest's logic of paradox, 4-valued systems like Dunn–Belnap's logic, the n-valued systems of Łukasiewicz and Post, and even infinitely-valued systems such as Łukasiewicz logic Ł [29] or Gödel–Dummett logic G [15].

Let us illustrate these many-valued semantics by taking a look at the definition of two operations intended as interpretations of the implication connective in Łukasiewicz and Gödel–Dummett logic respectively (for values $a, b \in [0, 1]$):

$$a \to^{\text{Ł}} b = \begin{cases} 1 & \text{if } a \leq b \\ 1{-}a{+}b & \text{otherwise} \end{cases} \qquad a \to^{\text{G}} b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$$

These examples showcase a typical feature of many-valued logics: the multiple values in the semantics do not form an arbitrary chaotic set, but they follow a prescribed order, in this case the standard order of the real numbers in $[0, 1]$. Then, we may say that, if $a \leq b$, then b accounts for propositions that are *at least as true* as those that are given the value a. The greatest value of the set, the number 1, is then taken as representing *full truth.* This allows us to argue that, despite the complexity of the many-valued semantics, the behavior of implication still bears a strong resemblance to some aspects of classical implication. That is to say, the two mentioned examples still follow the guiding idea of *truth preservation*, which now can be formulated as: if an implication is *fully true* (i.e. takes value 1), its consequent cannot be less true than its antecedent.

As in the case of classical logic, the algebraic (this time many-valued) operations for all connectives present in the language of a many-valued logic give rise to *evaluations,* i.e. mappings assigning to each formula, in a structure-preserving way, an element of the set of truth-values. Evaluations are then essential for extending the classical definition of *semantical consequence*. For instance, one defines $\Gamma \vDash_{\text{Ł}} \varphi$ as: each evaluation in the $[0, 1]$-valued semantics that gives value 1 to all formulas in Γ must also give value 1 to φ (and analogously for $\Gamma \vDash_{\text{G}} \varphi$). It is, hence, preservation of the notion of full truth given by the value 1. Similar truth-preserving definitions can be given in general for any many-valued logic, hence giving a multitude of alternative semantical accounts of correct reasoning.

On the other hand, many-valued logics also enjoy the classical repertoire of proof systems by which they are endowed with a syntactical notion of inference. Naturally, a fundamental result in the mathematical study of each many-valued logic is the corresponding completeness theorem that guarantees the perfect link between an intended notion of semantical consequence and a given syntactical proof system.

More recently, the field of algebraic logic has developed a paradigm in which *most* systems of non-classical logics can be seen as many-valued logics, because they are given a semantics in terms of algebras with more than two truth-values. From this point of view, many-valued logics encompass wide well-studied families of logical systems such as relevance logics, intuitionistic logic and its extensions, and even the family of substructural logics on which we will comment next (see e.g. [20]).

2.2 Substructural logics

Classical logic can be presented, among other syntactical options, by a proof system based on *sequents* introduced by Gentzen [21, 22]. It is constituted by

- *logical* rules governing the behavior of connectives and
- *structural* rules which do not refer to any particular connective.

Logics lacking some of these structural rules (most importantly, those known as *exchange, weakening, contraction,* and *associativity*) are studied in the literature under the name *substructural logics*. As explained in the monographs [32,39,43] they form a huge class of non-classical logics including: relevant logics (amenable to deal with the aforementioned unintuitive behavior of classical material implication), linear logics (introduced in theoretical computer science as resource-aware systems) or Lambek calculi (introduced in linguistics to deal with grammatical categories in formal and natural languages).

Although they have been proposed in terms of largely unrelated motivations and have been the subject of many independent studies, in recent years an increasing number of authors have followed the systematic unifying approach of algebraic logic that allows one to see substructural logics as a specific kind of many-valued logics. Indeed, a long stream of purely algebraic papers has concentrated on the algebraic semantics of substructural logics, based on *residuated lattices*, and have resulted in a deep knowledge of these logics (see e.g. the monograph [20]).

One of the advantages of residuated lattices is the presence of their lattice order, which allows us to keep the aforementioned idea of values ordered according to their truth, that is, for elements a and b of a residuated lattice $\boldsymbol{A}$ with order $\leq$, we say that b is at least as true as a whenever $a \leq b$. Moreover, the propositional language of substructural logics typically includes a conjunction connective & and constant symbol $\bar{1}$ respectively interpreted in $\boldsymbol{A}$ by a binary operation $\&^{\boldsymbol{A}}$ and its neutral element $\bar{1}^{\boldsymbol{A}}$. The latter can be used to define the following set of *designated elements*:

$$F = \{a \in A \mid a \geq \bar{1}^{\boldsymbol{A}}\}$$

which accounts for the *full truth* that has to be preserved in semantical consequences (hence, contrary to the previously seen examples of many-valued logics, now there may be many truth-values which are considered fully true and $\bar{1}^{\boldsymbol{A}}$ is just the least of them):

$$\begin{array}{lll} \Gamma \models_{\boldsymbol{A}} \varphi & \text{if and only if} & \text{for each evaluation } v \text{ in } \boldsymbol{A}, \\ & & \text{if } v(\gamma) \in F \text{ for each } \gamma \in \Gamma \text{, then } v(\varphi) \in F. \end{array}$$

Focusing again on the behavior of implications, let us point out that the semantical counterpart of an implication $\to$ in $\boldsymbol{A}$ is a binary operation $\to^{\boldsymbol{A}}$ satisfying the following *residuation* property with respect to the operation $\&^{\boldsymbol{A}}$ for any elements $a, b, c \in A$:[1]

$$a \&^{\boldsymbol{A}} c \leq b \qquad \text{if and only if} \qquad c \leq a \to^{\boldsymbol{A}} b.$$

[1]Since we do not assume $\&^{\boldsymbol{A}}$ to be commutative, the expression below is, strictly speaking, only a half of the residuation property.

Taking $c = \bar{1}^{\boldsymbol{A}}$, we obtain a crucial relation between the lattice order, the implication operation, and the set of designated elements:

$$a \leq b \qquad \text{if and only if} \qquad a \rightarrow^{\boldsymbol{A}} b \in F,$$

which captures the idea that an implication is fully true exactly when the consequent is truer than the antecedent.

Structural rules have natural interpretations, in residuated lattices, as properties of $\&^{\boldsymbol{A}}$: in particular, exchange makes it commutative, weakening identifies its neutral element $\bar{1}^{\boldsymbol{A}}$ with the greatest element of the lattice order $\leq$, and contraction (together with weakening) makes it idempotent.

2.3 Fuzzy logics

In mathematics, all terms are assumed to be precise and well-defined, in the sense that every property is expected to yield a perfect division between the objects which satisfy it and those which do not. That is why classical logic is well-tailored to model reasoning in usual mathematical practice. However, as soon as one considers non-mathematical contexts, one immediately has to deal with vague predicates (such as *old*, *tall*, or *warm*) for which it is not possible to establish such a clear division. Fuzzy logics have been proposed as non-classical systems for dealing with vagueness [44]. They are based on two main principles:

- The truth of vague propositions is a matter of degree.
- Degrees of truth must be comparable.

Inside the family of many-valued logics one can easily find good systems that satisfy these principles. The most typical examples are $[0, 1]$-valued logics (like Ł and G).

Historically, fuzzy logic emerged from fuzzy set theory (first proposed in 1965 by Lotfi Zadeh [51]). Such theory became extremely popular in computer science and engineering giving rise to a whole area of research with uncountable applications (see e.g. [40, 42]), but it lacked a focus on the usual aspects studied by logicians, e.g. formal language, semantics, proof systems, analysis of arguments, etc.

To remedy this shortcoming, at the beginning of the 1990s, based on earlier works [23, 30, 31, 33–35, 47, 48] the Czech mathematician Petr Hájek became the leader of a tour de force to provide solid logical foundations for fuzzy logic. In his approach, that soon was followed by numerous researchers in mathematical logic, fuzzy logics were taken as non-classical systems with a many-valued semantics that reflects the principle of comparability of degrees of truth. He achieved this by restricting the behavior of implication with the following *prelinearity axiom*:

$$(\varphi \to \psi) \vee (\psi \to \varphi).$$

Hájek's monograph [24] studied fuzzy logics with the tools of algebraic logic and gave birth to a whole new subdiscipline of mathematical logic, called *mathematical fuzzy logic* (MFL), specialized in the study of this family of many-valued logics.

Interestingly enough, many fuzzy logics have been identified as a particular kind of (axiomatic extensions of) substructural logics whose algebraic semantics is generated by linearly ordered algebras [16] (coherently with the principle that all degrees of truth must be comparable). The last two decades have witnessed a great development of MFL (see it presented in the handbook series [5–7]) and a proliferation of fuzzy logics with diverse properties but always keeping completeness with respect to linearly ordered algebras.

3 Abstract algebraic logic

The growing multiplicity of logics certainly calls for a uniform general treatment. A natural candidate for such a general theory is algebraic logic, the branch of mathematical logic that studies logical systems by giving them a semantics based on some algebraic structures. As mentioned above, this branch has served as a unifying approach to deal with the increasingly populated landscape of non-classical logics and has developed a variety of techniques which have been fruitfully applied to many families of logics, including those we have just seen.

In the last four decades, algebraic logic has evolved into a more abstract discipline, abstract algebraic logic (AAL), which aims at understanding the various ways in which a logical system (in an arbitrary language) can be endowed with an algebraic semantics. The pioneering works in this area are those from the Polish logic school [28, 38, 49, 50]. Later, the theory was thoroughly developed and systematized mostly by Willem J. Blok, Janusz Czelakowski, Josep Maria Font, Ramon Jansana, Don L. Pigozzi, and James G. Raftery [2, 14, 17, 18, 37].

By understanding the deep connection between logics and their algebraic semantics, AAL has provided a corpus of results that allows one to study properties of the logical systems by means of (equivalent) algebraic properties of their semantics.

Also, AAL has led to a finer analysis of the role of the connectives of classical logic, identifying their essential properties, and thus suggesting possible generalizations of these connectives (in non-classical logics) still retaining the essential function(s) they play in classical logic. Notable examples of this approach are the extensive studies on equivalence (or biconditional) connectives in the works we have just mentioned. Indeed, the Lindenbaum–Tarski proof of completeness of the classical propositional calculus, based on the fact that the equivalence connective defines a congruence on the algebra of formulas, has been extended to broad classes of logics by considering a suitable generalized notion of equivalence. That is, equivalence can be taken as a (possibly parameterized, possibly infinite) set of formulas in two variables satisfying certain simple properties. This approach gave rise to the important class of protoalgebraic logics [1] characterized by the presence of a generalized equivalence.

Similarly, there have been works focusing on suitable notions of conjunction (e.g. [26]), disjunction (e.g. [10, 14]), negation (e.g. [36]), and implication (e.g. [3, 13, 25] and, especially, Rasiowa's book of *implicative logics* [38]).

4 Logics and implication

In logical consequence the truth of a set of premises is 'transmitted' to a conclusion. We have seen that both in classical and in many non-classical logics this idea guides the definition of both the semantics (algebraic operations) and the inference rule (*modus ponens*) of implication. In many logics, including classical, the relation between logical consequence and implication is very straightforward thanks to the *deduction theorem*:

$$\varphi \vdash \psi \qquad \text{if and only if} \qquad \vdash \varphi \to \psi,$$

that is, the implication connective internalizes logical consequence, which is one of the reasons why implication may be seen as the main logical connective.

Moreover, algebras of fuzzy logics (and, more generally, of substructural logics) have an order relation naturally determined by the implication operation. In the most well-known setting, this can be described in the following way:

$$a \leq b \qquad \text{if and only if} \qquad a \to^{\boldsymbol{A}} b = \bar{1}^{\boldsymbol{A}}.$$

More generally, as we have seen in substructural logics, one may need to consider not just the value $\bar{1}^{\boldsymbol{A}}$, but the set F of *designated elements* in the algebra, which allows one to define the order in terms of the implication as

$$a \leq b \qquad \text{if and only if} \qquad a \to^{\boldsymbol{A}} b \in F.$$

This easy, and very well-known, observation inspired the applicant, Petr Cintula, to start developing in 2005 a general framework for all logics with this property. Generalizing Rasiowa's implicative logics (which still were restricted to logics with a semantics with a greatest degree of truth which, moreover, is the only designated value), the paper [4] introduced the class of *weakly implicative logics* as those with a binary connective $\Rightarrow$ that enjoys what we consider the minimal basic requirements any implication connective should satisfy:[2]

$$\varphi \Rightarrow \varphi$$
$$\varphi, \varphi \Rightarrow \psi \blacktriangleright \psi$$
$$\varphi \Rightarrow \psi, \psi \Rightarrow \chi \blacktriangleright \varphi \Rightarrow \chi$$
$$\varphi \Rightarrow \psi, \psi \Rightarrow \varphi \blacktriangleright c(\alpha_1, \dots \alpha_i, \varphi, \dots \alpha_n) \Rightarrow c(\alpha_1, \dots \alpha_i, \psi, \dots \alpha_n)$$
$$\text{for each } n\text{-ary connective and each } 0 \leq i < n.$$

These conditions are enough to guarantee that the connective $\Rightarrow$ defines an order relation in the algebraic semantics in the way we have just seen. Moreover, the framework allows one to capture a precise mathematical rendering of the informal notion of *fuzzy logics* as those that are complete with respect to their linearly ordered (by the order given by the implication) algebraic models. These systems and their abstract theory have been studied as *semilinear logics* as an AAL-style foundation of MFL in [9].

This approach has been later extended and developed in detail in a series of papers [8, 11, 12] as the theory of *weakly p-implicational logics*, in which implications are taken as connectives defined by (possibly infinite and parameterized) sets of formulas. The requirements on such generalized connectives are indeed very weak and encompass a very broad class of logics. Actually, p-implicational logics turn out to be an alternative presentation of the *protoalgebraic logics*, a fundamental class of logics deeply studied in the core theory of AAL. However, the stress on generalized implications has allowed us to focus better on certain aspects of these logics.

Interestingly enough, most of the advantages of this implication-based general approach are already available at the level of weakly implicative logics. Indeed, this class already contains most of the prominent non-classical logics studied in the literature, since they almost always have a reasonable implication connective. In particular, weakly implicative logics provide a good framework to study fuzzy, substructural and many-valued logics.

[2]Unbeknownst to the applicant, these conditions had been singled out independently by different authors before. On the one hand, Richard Sylvan (née Francis Richard Routley) and Meyer already in their 1975 paper [41] argued on page 70 that these conditions are in a certain sense minimal sufficient properties for a connective to be a reasonable implication. They noted that equivalence is a special kind of weak implication and argued that what distinguishes implication is its interplay with other connectives, which is something that we study in detail in Chapter 4. On the other hand, in 1980 Jacek K. Kabziński, in an attempt to capture natural notions of (pre)ordered matrix models by syntactical means, also defined the notion of weakly implicative logics, with different notation and terminology, in [27], building on previous works by Suszko [45, 46].

A critical point of any general theory is the choice of its level of generality. At the time of its publication, Rasiowa's monograph on implicative logics [38] had struck an excellent level of generality, broad enough to cover most of the research being done at the time, and yet not too far as to become too abstract and difficult to understand and use. The subsequent development of non-classical logics, however, made it obsolete. Many important logics, studied in theoretical papers and sometimes used in applications to other areas, did not fit anymore in the class of implicative logics with its rather narrow defining restrictions. The present dissertation intends to remedy this shortcoming by presenting the theory of weakly implicative logics as a new framework that can have today the same advantages that Rasiowa's class used to have.

5 Content and structure of the dissertation

Besides the introductory chapter, the dissertaion is structured in six additional chapters and one Appendix. The latter presents the basic preliminary mathematical notions and results used throughout the text. It starts with elementary notions of basic set theory, orders, closure systems, and lattices; after that it presents the basics of universal algebra and is concluded by recalling some notable classes of algebras related to non-classical logics (modal, Heyting, Gödel, and MV-algebras). Let us briefly describe the content of the main chapters.

5.1 Chapter 2: Weakly implicative logics

This chapter is the real beginning of our story and hence it is devoted to presenting its main object of study: the class of *weakly implicative logics.*

We start by introducing basic syntactical notions (variables, connectives, formulas, Hilbert-style proof systems, etc.) and giving a purely syntactical definition of logics as mathematical objects (namely, as structural consequence relations).

After testing the definition with three extreme, mostly uninteresting examples, we immediately present some of the most important logics that one can find in the literature and that will accompany the reader throughout the text: classical logic, intuitionistic logic, Łukasiewicz logic (both its finitary and infinitary version), Gödel-Dummett logic, and the implicational logics known as BCI, $A_{\to}$, and BCK. We conclude the syntactical part of the chapter with a brief study of important metalogical properties of these logics such as (variants of) the deduction theorem and the proof by cases property; these properties are later, in Chapters 4 and 5, studied in a much more abstract setting.

Next, we start introducing basic *semantical* notions. The fundamental one is that of a *logical matrix*, which is an arbitrary algebra equipped with a set of designated truth-values, called the *filter* of the matrix. Any logic can be assigned a class of logical matrices, whose members we call *models* of the logic, which can be shown to provide a complete semantics, albeit a very uninformative one with a very loose connection to the logic in question. In order to obtain a more meaningful semantics, we introduce the notion of Leibniz congruence of a given matrix, which allows us to define, for any given logic, a class of *reduced* models which provide a more refined complete semantics and are more tightly related to the logic in question.

Finally, we give a syntactical definition of the class of *weakly implicative logics* and show that these are exactly the logics where the Leibniz congruence admits a simple description using the *implication* connective. This connective can also be used to define another fundamental notion of the dissertation: the *matrix order* in the reduced models of weakly implicative logics. The chapter is concluded by introducing and exploring the class *algebraically implicative logics*, in which one can work just with algebras instead of matrices.

5.2 Chapter 3: Completeness properties

This chapter presents the foundations of the theory of logical matrices with a special focus on the question of which classes of matrices provide a complete semantics for a given logic. We identify three kinds of completeness based on how we restrict the cardinality of the sets of premises: we distinguish *strong* completeness, where there is no restriction, *finite strong* completeness, where we restrict ourselves to finite sets of premises, and weak *completeness,* where we disregard premises altogether and speak about theorems and tautologies only. Throughout the chapter we introduce increasingly complex model-theoretic constructions on (classes of) matrices and use them to characterize not only the completeness properties but also other important notions.

We start by introducing submatrices, homomorphisms and direct products of matrices. We use these tools to improve our understanding of the Leibniz congruence and obtain an important semantical characterization of the notion of conservative expansions and of the class of algebraically implicative logics. Our next tools are the *subdirect* products and subdirectly irreducible matrices. We show that each finitary logic is strongly complete with respect to such matrices, in particular recovering the completeness of classical logic with respect to the two-element Boolean algebra. Finally, the most complex construction we use are the *filtered products*, in particular countably-filtered products and ultraproducts, which we use to give a purely semantical characterization of finitary logics and completeness properties.

5.3 Chapter 4: On lattice and residuated connectives

Clearly, not only implication, but also other logical connectives are crucial for the theory and the applications of particular logics. Hence, this chapter is devoted to the study of two groups of important connectives: lattice ($\wedge$, $\vee$, $\top$, $\bot$) and residuated connectives ($\&$, $\rightsquigarrow$, $\bar{1}$, $\bar{0}$):

- The lattice connectives allow us to express properties of the matrix order in the models of the logic in question: $\wedge$ and $\vee$ are binary connectives whose intended interpretation is, respectively, the *infimum* and the *supremum* of this order, whereas $\top$ and $\bot$ are truth-constants that are intended to be interpreted by respectively the *greatest* and the *least* element of the order.
- The *residuated conjunction* $\&$ is a binary connective intended to work as a means to aggregate premises in chains of nested implications. This conjunction is not assumed, in general, to be commutative (neither associative nor idempotent). In logics in which $\&$ is commutative, we can switch the order of premises in nested implications. Otherwise, we can add the binary connective $\rightsquigarrow$ (*co-implication*) which allows us to do the switch at the price of replacing the original inner implication by $\rightsquigarrow$. Algebraically, the relation between $\&$, $\to$, and $\rightsquigarrow$ is described by the *residuation property*:

  $$x \leq y \Rightarrow^{\boldsymbol{A}} z \qquad \text{iff} \qquad y \mathbin{\&^{\boldsymbol{A}}} x \leq z \qquad \text{iff} \qquad y \leq x \rightsquigarrow^{\boldsymbol{A}} z.$$

  The truth-constant $\bar{1}$ is intended to stand for the *protounit*, i.e. the minimum element of the filter of reduced models, and in certain logics it becomes the *unit*, i.e. the neutral element of $\&$. Finally, the truth-constant $\bar{0}$ is introduced without any intended algebraic interpretation, just as a device used in the literature to define negation connectives.

We start by exploring the logical and algebraic properties of these connectives which we introduce using Hilbert-style rules which enforce the expected semantical behavior and study consequences of their presence in a logic.

After that, we introduce two minimal logics containing certain collections of these connectives: Lambek logic LL (the least weakly implicative logic where all the residuated connectives have the minimal intended behavior) and the logic SL (the least expansion of LL where also the lattice connectives have the minimal intended behavior and furthermore the truth constant $\bar{1}$ is the unit). We axiomatize them by means of strongly separable Hilbert-style calculi and describe their classes of reduced models and show that they admit regular completions. These two logics also serve as bases for our study of *substructural logics*. Indeed, we define them as the class of weakly implicative logics expanding the implicative fragment of LL and then we build a hierarchy of substructural logics expanding SL, which contains the most prominent members of this family.

The rest of the chapter is devoted to the study of deduction theorems in substructural logics, capitalizing on the presence of implication and residuated conjunction. We introduce a general notion of *implicational deduction theorem* and provide a characterization in terms of the existence of a presentation that only has one binary rule (*modus ponens*) and unary rules of a certain simple form. As interesting by-products, for any logic enjoying such deduction theorem, we obtain a description of filter generation (in its reduced models) and a construction technique for a new form of generalized disjunction connective (given by a set of formulas) with the proof by cases property.

5.4 Chapter 5: Generalized disjunctions

The general kind of disjunctions enjoying the proof by cases property obtained for substructural logics in the previous chapter (given by a set of formulas instead of a single disjunction connective) motivates the abstract study of generalized disjunctions that we present in the fifth chapter.

First, we introduce several forms of proof by cases property and corresponding generalized disjunctions, which we usually denote as $\triangledown$, yielding a hierarchy of logics that we illustrate and separate with suitable examples. The we provide several (groups of) characterizations of generalized disjunctions:

- The first characterization is based on the notion of $\triangledown$-form of a rule, that is the rule obtained from the original rule by disjuncting its premises and conclusion with an arbitrary formula. This notion allows us to characterize logics with a strong p-disjunction $\triangledown$ as those closed under the formation of $\triangledown$-forms of its rules. This characterization is then used to study the preservation of the proof by cases property in expansions and to prove its transfer to the general matrix semantics.
- The second group of characterizations is based on various generalized distributivity properties of the lattice of filters.
- The third characterization is based on $\triangledown$-prime filters, a generalization of the well-known notion of prime filter for classical and intuitionistic logic and its relation to the intersection-prime filters.

Finally, we use generalized disjunctions in order to: (1) obtain some axiomatizations of interesting extensions of a given logic, (2) introduce a symmetric notion of consequence relation with a disjunctive reading on the right-hand side, (3) improve some of the characterizations of completeness properties seen in Chapter 3, and, finally, (4) study completeness with respect to finite matrices and matrices with $\triangledown$-prime filters.

5.5 Chapter 6: Semilinear logics

This chapter focuses on the other family that motivated the general study of logics with implication: *semilinear* logics, defined as logics complete with respect to *linearly ordered* reduced matrices.

We start by formulating and proving useful characterizations of semilinear logics in terms of linear filters, a syntactical metarule akin to the proof by cases property, and the coincidence of finitely subdirectly irreducible and linearly ordered reduced matrices. We use these characterizations to show which of the examples of weakly implicative logics considered in the previous chapters are actually semilinear logics and prove which of them are not semilinear with respect to any possible implication. Then we study the problem of, given an arbitrary weakly implicative logic (in particular, given a substructural logic), finding its least semilinear extension. We also use the presence of disjunction and the results obtained in the previous chapter to prove better characterizations of semilinearity leading to axiomatizations of the least semilinear extension of a given logic.

We conclude the chapter by exploring the completeness properties with respect to the subclass of linear models in which the order is dense and discuss its relation with completeness with respect to reduced matrices defined over the rational and the real unit intervals.

5.6 Chapter 7: First-order predicate logics

The last chapter gives a short introduction to the study of first-order predicate logics built over weakly implicative logics. We follow a semantics-first approach in which we start from semantically defined predicate logics and then propose suitable Hilbert-style axiomatizations and prove corresponding completeness theorems by following non-trivial generalizations of Henkin's proof of completeness of classical first-order logic. More precisely, to introduce a general semantics of predicate models we utilize the fact that any reduced model of a given weakly implicative logic is ordered and define the truth-value of a universal (resp. existentially) quantified formula as the infimum (resp. supremum) of the truth-values of its instances.

Then, for any given class of reduced models of the logic in question, we define a corresponding consequence relation on predicate formulas. We focus on three particular meaningful logics: the predicate logic of all reduced models, the subdirectly irreducible ones, and the restriction of the latter to *witnessed* predicate models in which quantifiers are actually computed as minima and maxima. We propose axiomatic systems for these three logics and, under certain conditions, prove the corresponding completeness theorems.

While the first completeness result is relatively straightforward and works in absolute generality, the other two require non-trivial modifications of Henkin's proof by making use of a suitable disjunction connective which needs to be added as a requirement for the propositional logic. As a by-product, we also obtain, for certain logics, a form of Skolemization. The relatively modest assumptions on the propositional side allow for a wide generalization of previous approaches and help to illuminate the 'essentially first-order' steps in the classical Henkin's proof.

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List of applicant's publications this dissertation is build on

This dissertation is the monograph:

- P. Cintula, C. Noguera. *Logic and Implication: An Introduction to the General Algebraic Study of Non-classical Logics.* Springer, 2021.

Besides numerous previously unpublished results the monograph on its almost 500 pages contains applicant's contributions published in the following papers:

- M. Bílková, P. Cintula, T. Lávička. Lindenbaum and pair extension lemma in infinitary logics. In Moss, De Queiroz, Martinez, eds: *Proceedings of WoLLIC'18,* vol. 10944 of LNCS, pp. 130–144, Springer, 2018.
- P. Cintula, C. Noguera. Implicational (semilinear) logics III: Completeness properties. *Archive for Mathematical Logic* 57:391–420, 2018.
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Resumé

The dissertation is the research monograph coauthored with Carles Noguera on the general algebraic study of non-classical logics. It offers a systematic study of weakly implicative logics, a class of logics covering a vast part of the landscape of non-classical logics studied in the literature, including the prominent families of substructural, fuzzy, relevant, and modal logics. Its main method is the abstract mathematical study of the relation between logics and their algebraic semantics, concentrating mainly on the role of particular connectives in this relationship. Needless to say, neither the algebraic method, nor the mentioned prominent families of logics are our invention. What makes our monograph unique is its focus on implication as the main connective, right balance between abstraction and usability of the presented approach, and its self-contained and didactic presentation which allows it to also serve as a textbook for:

- abstract algebraic logic,
- substructural and fuzzy logics, and
- the study of the role of implication and disjunction in logic.

What has led us to writing this monograph? We are researchers who have been studying substructural and fuzzy logics for years using mainly the tools of algebraic logic. Soon enough in this endeavor, we observed the existence of a great deal of repetition in the papers published on this topic. It was common to encounter articles that studied slightly different logics by repeating the same definitions and essentially obtaining the same results by means of analogous proofs. We felt it as an unnecessary ballast that was delaying the development of such logics and obscuring the reasons behind the main results. This was an area of science screaming for systematization through the development and application of uniform, general, and abstract methods. The need for such a systematization project brought us together.

Stemming from our background education, abstract algebraic logic presented itself as the ideal toolbox to rely on. It was a general theory applicable to all non-classical logics and it provided an abstract insight into the fundamental (meta)logical properties at play. However, the existing works in that area did not readily give us the desired answers.

Despite their many merits, these texts lived at a level of abstraction a little too far detached from our intended field of application. They were indeed great sources of knowledge and inspiration, but there was still a lot of work to be done in order to bring the theory closer to the characteristic particularities of substructural and fuzzy logics. Namely, we identified a number of properties

codified in the logical connectives (mainly in implication and disjunction) that make these families of logics unique and interesting, and we observed that these properties could be (needed to be!) studied with the methods of abstract algebraic logic.

These considerations led to an extensive series of papers which culminated in the present book, whose goal is to present a matured up-to-date theory which is powerful, general yet readily accessible, and keeps the right balance between abstraction and usability. Furthermore, we want to do that in a reasonably self-contained and didactic manner, starting from very elementary notions and building brick-by-brick a rather involved theory with a substantial number of results of various level of difficulty and abstraction.

Who is this monograph intended for? We intend to reach a fairly wide audience:

- students and scholars looking for an introduction to a general theory of non-classical logics and their algebraic semantics;
- experts in non-classical logics looking for particular results on their favorite logics;
- readers with background in mathematics, philosophy, computer science, or related areas, and an interest in formal reasoning systems that are sensitive to a number of intriguing phenomena (vagueness, graded predicates, constructivity, relevance, non-commutativity, non-associativity, resource-awareness, etc.); and
- teachers of logic and related fields, who may use parts of the book as supporting material for their courses on topics related to (abstract) algebraic logic, non-classical, substructural, and fuzzy logics.

How is this monograph structured? After the introduction (where we summarize the driving motivations behind this book and justify its main design choices), the second and third chapters develop the foundations of our approach on which the remainder of the book rests. Their content is based on deeply intertwined topics that are presented in the necessary logical order. The subsequent chapters, from the fourth one onwards, focus on advanced topics, and while these topics are to a large extent independent, many particular results and examples of one chapter are used in the subsequent ones to illustrate and deepen their results. Each chapter is concluded by a list of proposed exercises and a section putting its content into a wider, historical and mathematical context of research in relevant parts of logic. Finally there is an appendix containing the necessary elementary notions and facts from set theory, lattice theory, and universal algebra.